

Structure-preserving model reduction

From the formulation on manifolds to data-driven realizations

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Motivation: multi-query and real-time evaluation

repetitive evaluations of the model (*multi-query*)

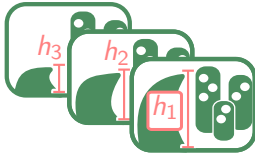


real-time evaluation
of the model



simulation

e.g. parameter study



optimization



(model-based) control



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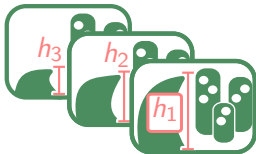


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Model might be *too slow* to be solved for *multi-query* or in *real-time* → MOR

Motivation: Hamiltonian Systems

Hamiltonian system: $\frac{d}{dt}\mathbf{x}(t; \boldsymbol{\mu}) = \mathbb{J}_{2N} \nabla_{\mathbf{x}} \mathcal{H}(\mathbf{x}(t; \boldsymbol{\mu}); \boldsymbol{\mu})$ with

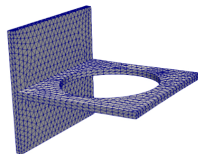
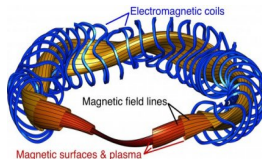
$$\mathcal{H}(\cdot; \boldsymbol{\mu}) : \mathbb{R}^{2N} \rightarrow \mathbb{R}, \quad \mathbb{J}_{2N} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \quad \mathbb{J}_{2N}^\top = -\mathbb{J}_{2N}.$$

Properties of Hamiltonian systems:

- closed systems
- energy-preserving

Possible applications fields are, e.g.,

- plasma physics
- planetary motion
- linear elasticity problems



Example: Structure-preserving MOR - Wave Equation

Linear 1D Wave Equation: $\Omega := [-0.5, 0.5]$, $t \in [0, 4]$, $\mu = 0.5$

$$\begin{aligned}\partial_{tt}^2 \varphi(t, \xi; \mu) &= \mu^2 \partial_{\xi\xi}^2 \varphi(t, \xi; \mu), & \text{on } I \times \Omega, \\ \varphi(0, \xi; \mu) &= \varphi_0(\xi; \mu), & \forall \xi \in \Omega, \\ \partial_t \varphi(0, \xi; \mu) &= -\mu \partial_{\xi} \varphi_0(\xi; \mu), & \forall \xi \in \Omega, \\ \varphi(t, \xi; \mu) &= 0, & \forall t \in I, \xi \in \{-0.5, 0.5\},\end{aligned}$$

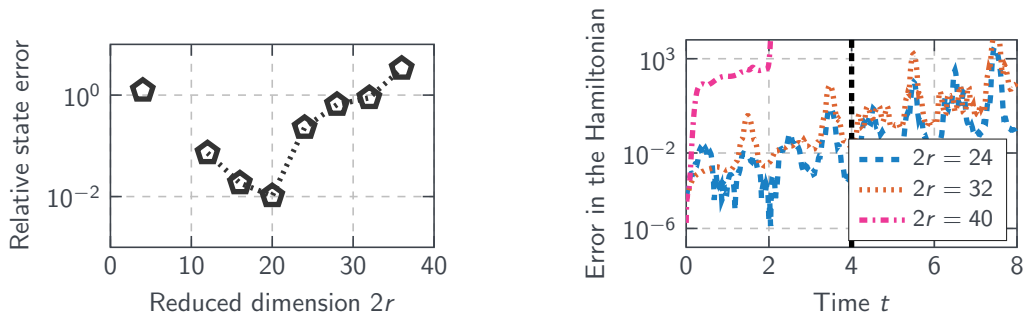
Discrete FOM model:

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{\mathbf{q}}(t) \\ \dot{\mathbf{p}}(t) \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix}, \quad \text{with } \mathbf{A} := \begin{pmatrix} \mathbf{0} & \mathbb{I}_n \\ \frac{1}{4} \mathbf{D}_{xx} & \mathbf{0} \end{pmatrix}$$

- Time discretization: **implicit midpoint** with $n_t = 4000$,
- State discretization: $2n = 4096$.

What can go wrong? Illustrative Example

Numerical Results for Hamiltonian system without structure-preserving MOR:



Setting in this talk: Projection-based MOR

- model order reduction (MOR) for dynamical systems
 - given: parametric high-dimensional system of ODEs

$$\frac{d}{dt}\mathbf{x}(t; \boldsymbol{\mu}) = \mathbf{f}(\mathbf{x}(t; \boldsymbol{\mu}); \boldsymbol{\mu}) \in \mathbb{R}^N, \quad (t; \boldsymbol{\mu}) \in \mathcal{I} \times \mathcal{P} \subset \mathbb{R}^{n_p+1}, \quad N \gg 1$$

- determine efficient surrogate model for multi-query (optimization, parameter studies, UQ)
- projection-based MOR

full-order model (FOM) $\xrightarrow{\text{projection}}$ reduced-order model (ROM)

- In this talk:
 - 1 geometric framework for MOR on manifolds
 - 2 how to preserve structure, e.g., Hamiltonian systems?

Outlook:

1. Introduction and Challenges of classical MOR
2. Extending to MOR on Manifolds
3. Data-driven Realization
4. Numerical Experiments for Hamiltonian systems
 - ① Autoencoder
 - ② Quadratic Embedding
5. Conclusion

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(Petrov–Galerkin) projection-based MOR

high-dimensional ODE system $\frac{d}{dt}\mathbf{x}(t; \mu) = \mathbf{f}(\mathbf{x}(t; \mu), t; \mu) \in \mathbb{R}^N$

- ① approximation in low-dim subspace of reduced dimension $n \ll N$

$$\mathbf{x}(t; \mu) \approx \mathbf{V}\check{\mathbf{x}}(t; \mu), \quad \begin{array}{l} \check{\mathbf{x}}(t; \mu) \in \mathbb{R}^n : \text{reduced state,} \\ \mathbf{V} \in \mathbb{R}^{N \times n} : \text{reduced-order basis matrix} \end{array}$$

residual: $\mathbf{r}(t; \mu) = \mathbf{V} \frac{d}{dt}\check{\mathbf{x}}(t; \mu) - \mathbf{f}(\mathbf{V}\check{\mathbf{x}}(t; \mu), t) \in \mathbb{R}^N$

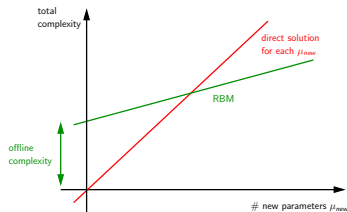
- ② projection of the residual onto a n -dimensional subspace $\mathcal{W} = \text{colspan}(\mathbf{W})$ for $\mathbf{W} \in \mathbb{R}^{N \times n}$

$$\mathbf{W}^T \mathbf{r}(t; \mu) \stackrel{!}{=} \mathbf{0}_{n \times 1} \Leftrightarrow \underbrace{\mathbf{W}^T \mathbf{V} \frac{d}{dt}\check{\mathbf{x}}(t; \mu) - \mathbf{W}^T \mathbf{f}(\mathbf{V}\check{\mathbf{x}}, t; \mu)}_{\text{reduced system}} \in \mathbb{R}^n$$

When to use classical MOR ... and when not

Consequences:

- reduced approximation can only be as good as truth approximation,
- complexity-offset by offline phase $\rightsquigarrow$ MOR (only) reasonable, if
 - many evaluations required ('multi-query context')
 - very fast or limited evaluations required ('realtime context')



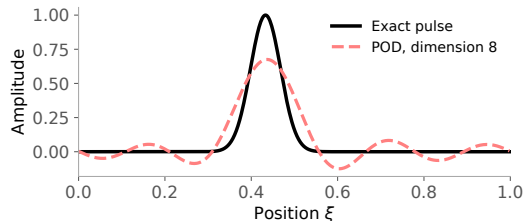
Challenges:

- **Challenge 1:** Benchmark: Kolmogorov N -width

$$d_n(\mathcal{P}) := \inf_{V_n; \dim(V_n)=n} \sup_{\mu \in \mathcal{P}} \inf_{\check{\mathbf{v}} \in V_n} \|\mathbf{x}(\mu) - \check{\mathbf{v}}\|$$

- **Challenge 2:** Reduced order model might be unphysical

Moving pulse: A fixed linear space can require many modes



Limitation: Kolmogorov n -width

$$d_n(\mathcal{P}) := \inf_{V_n; \dim(V_n)=n} \sup_{\mu \in \mathcal{P}} \inf_{\check{\mathbf{v}} \in V_n} \|\mathbf{x}(\mu) - \check{\mathbf{v}}\|$$

$\leadsto$ Slow decay for e.g.,

- Linear transport $d_n(\mathcal{P}) \geq \frac{1}{2} n^{-1/2}$ (Ohlberger, Rave, 2016)
- Wave equations $d_n(\mathcal{P}) \geq \frac{1}{4} n^{-1/2}$ (Greif, Urban, 2019)

We are not the only ones...

Shifted / transformed bases

Reiss, Schulze, Sesterhenn, Mehrmann (2018), Welper (2020), Black, Schulze, Unger (2020, 2021), ...

(Rational) Polynomial manifolds

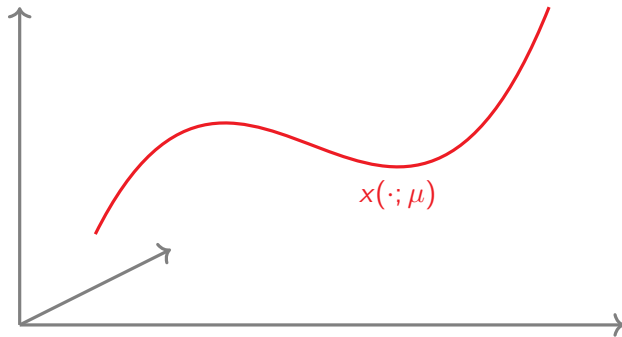
Barnett, Farhat (2022), Benner, Goyal, Heiland, Pontes Duff (2023), Geelen, Wright, Willcox (2023), Issan, Kramer (2023), Sharma, Mu, Buchfink, Geelen, Glas, Kramer (2023), Buchfink, Glas, Haasdonk (2024), Klein, Sanderse, Costa, Pecnik, Henkes (2025), ...

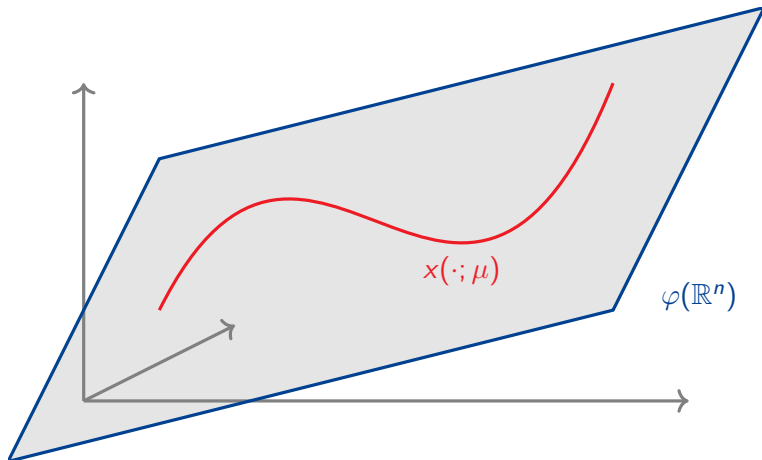
Autoencoders + MOR

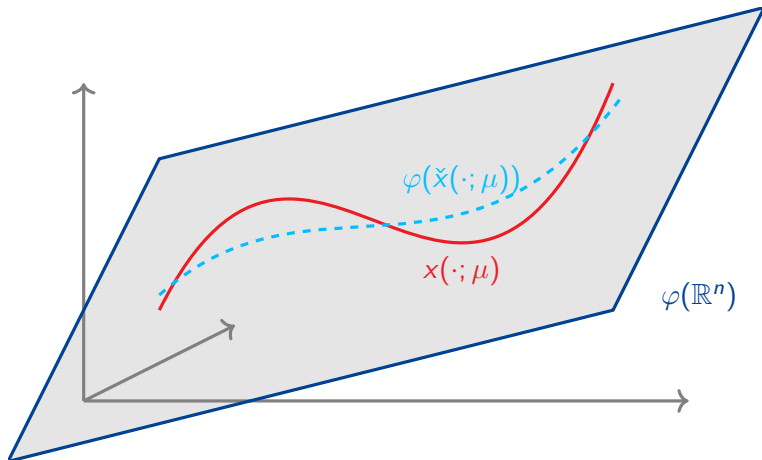
Kashima (2016), Hartman, Mestha (2017), Lee, Carlberg (2020), Kim, Choi, Wideman, Zohdi (2020), Fresca, Dedé, Manzoni (2021), Otto, Macchio, Rowley (2022), Buchfink, Glas, Haasdonk (2023), Brantner, Kraus (2023), Yildiz, Goyal, Bendokat, Benner (2024), Côte, Franck, Navoret, Steimer, Vigon (2025), Yildiz, Janik, Benner (2026), Friedl, Jaquier, Liao, Kragic (2026), Chen, Guo, Tang, Zhong (2026), Feng, Tang, Xie, Zhang, Zhu (2026), ...

and more...

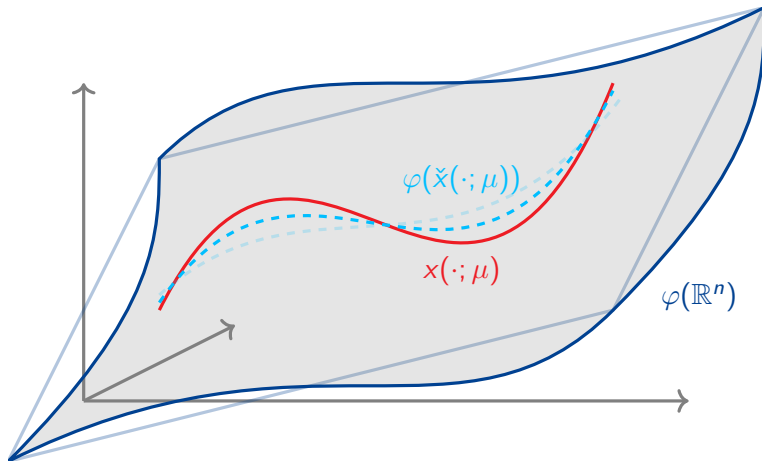
Registration methods, Neural Galerkin, Lifting methods, Dynamical low-rank methods, MOR on Spectral Submanifolds, ...







Linear $\rightarrow$ nonlinear



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Differential equations on manifolds - the FOM

The ingredients:

- a smooth N -dimensional manifold $\mathcal{M}$ (replacing the vector space $\mathbb{R}^N$)

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$$\begin{cases} \frac{d}{dt}\gamma|_t = X(\mu)|_{\gamma(t)} \in T_{\gamma(t)}\mathcal{M}, \\ \gamma(t_0) = \gamma_0 \in \mathcal{M} \end{cases}$$

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Important observation:

- $\gamma(t) \in \mathcal{M}$ vs. $\frac{d}{dt}\gamma|_t \in T_{\gamma(t)}\mathcal{M}$

MOR on manifolds

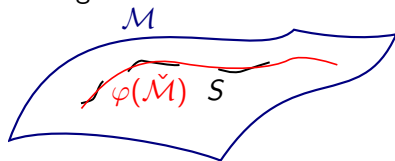
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 $\rightsquigarrow \varphi: \check{\mathcal{M}} \rightarrow \mathcal{M}$ smooth embedding

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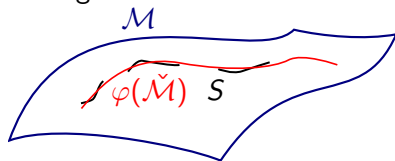
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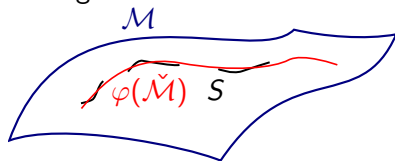


- A **reduction map** R —
smooth $R: T\mathcal{M} \rightarrow T\check{\mathcal{M}}$ satisfying $R \circ d\varphi = \text{Id}_{T\check{\mathcal{M}}}$

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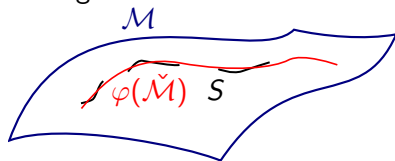


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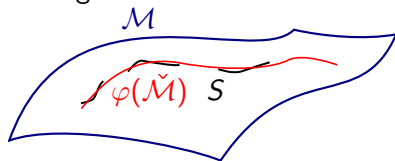
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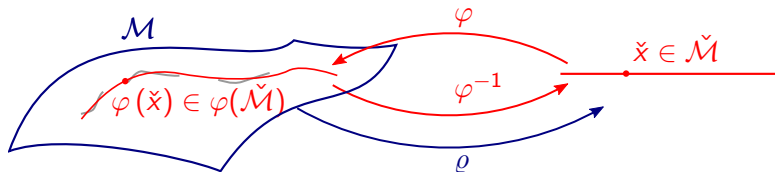


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Illustration: Point Reduction

- mapping points: Find **point reduction** $\varrho: \mathcal{M} \rightarrow \check{\mathcal{M}}$



- s.t. the **point projection property** holds

$$\varrho \circ \varphi = \text{id}_{\check{\mathcal{M}}}$$

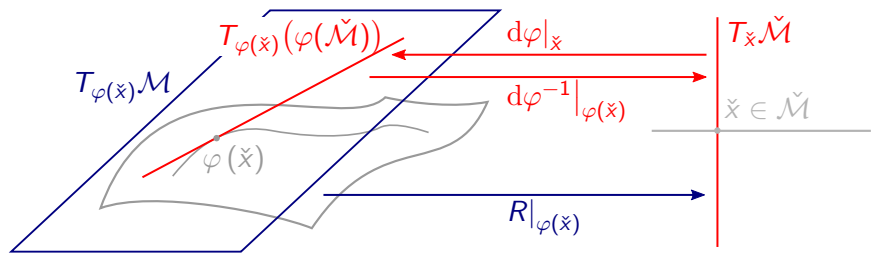
- Then

$$(\varphi \circ \varrho)^2 = \varphi \circ \varrho \circ \varphi \circ \varrho = \varphi \circ \varrho$$

is nonlinear projection

Illustration: Tangent Reduction

- mapping tangents at $\check{x} \in \check{\mathcal{M}}$ and $\varphi(\check{x}) \in \varphi(\check{\mathcal{M}})$
- Find tangent reduction $R|_x : T_x \mathcal{M} \rightarrow T_{\varphi(m)} \check{\mathcal{M}}$



- s.t. the tangent projection property holds

$$R|_{\varphi(\check{x})} \circ d\varphi|_{\check{x}} = \text{id}_{T_{\check{x}}\check{\mathcal{M}}} \quad \text{for all } \check{x} \in \check{\mathcal{M}},$$

Reduced order model (ROM)

Assume given:

- reduced manifold $\check{\mathcal{M}}$ with dimension $n \ll N$
- smooth embedding $\varphi: \check{\mathcal{M}} \rightarrow \mathcal{M}$
- reduction map $R(m, v) = (\rho(m), R|_m(v))$

The **reduced order model**:

$$\begin{cases} \frac{d}{dt}\check{\gamma}|_t = R|_{\varphi(\check{\gamma}(t))}(X(\mu)|_{(\varphi \circ \check{\gamma})(t)}) \in T_{\check{\gamma}(t)}\check{\mathcal{M}} \\ \check{\gamma}(t_0) = \rho(\gamma_0) \in \check{\mathcal{M}} \end{cases}$$

The MOR Workflow

(i) Approximation

Find a **reduced manifold** $\check{\mathcal{M}}$ and a **smooth embedding** $\varphi: \check{\mathcal{M}} \rightarrow \mathcal{M}$ such that

$$\varphi(\check{\mathcal{M}}) \approx S(\mathbb{T}; \mathcal{P}) \subseteq \mathcal{M}.$$

(ii) Reduction

Construct a **reduction map** $R: T\mathcal{M} \rightarrow T\check{\mathcal{M}}$ for φ and the associated ROM.

(iii) Reconstruction

Solve the ROM for $\check{\gamma}$ and approximate the solution γ via $\varphi(\check{\gamma})$

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Open questions: (addressed now)

- $\rightsquigarrow$ 1. How to construct **reduction maps**?
- $\rightsquigarrow$ 2. What about **structure**?

Theorem (Buchfink, G. Haasdonk, Unger, 2024)

Under the described setting let $\mu \in \mathbb{P}$ and assume $\gamma(t; \mu) \in \varphi(\check{\mathcal{M}})$ for all $t \in \mathbb{T}$. Then

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Standing assumption

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Observation: $\varphi: \check{\mathcal{M}} \rightarrow \varphi(\check{\mathcal{M}})$ is bijective.

$\leadsto$ construct smooth $\rho: \mathcal{M} \rightarrow \check{\mathcal{M}}$ such that $\rho|_{\varphi(\check{\mathcal{M}})} = \varphi^{-1}$, i.e., $\rho \circ \varphi = \text{Id}_{\check{\mathcal{M}}}$

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Example: Linear-subspace MOR

$$\rho(m) = W^\top m, \quad R|_m(v) = W^\top v, \quad \varphi(\check{m}) = V\check{m} \quad \text{with} \quad W^\top V = I_n$$

Manifold endowed with additional structure

Assume structure already at the FOM level:

- 1 smooth manifold $\mathcal{M}$ endowed with
- 2 nondegenerate $(0,2)$ -tensor field $\tau \in \Gamma(T^{(0,2)}(T\mathcal{M}))$

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Nondegeneracy of τ allows to formulate

- the inverse $(2,0)$ -tensor field $\tau^{-1} \in \Gamma(T^{(2,0)}(T\mathcal{M}))$
- an isomorphism between the tangent and the cotangent bundle through
- musical isomorphisms

$$b_\tau \in \mathcal{C}^\infty(T\mathcal{M}, T^*\mathcal{M}), \quad \sharp_\tau \in \mathcal{C}^\infty(T^*\mathcal{M}, T\mathcal{M}),$$

which are inverses of each other, i.e.,

$$\sharp_\tau \circ b_\tau \equiv \text{id}_{T\mathcal{M}}.$$

Additional structure: Generalized Manifold Galerkin (GMG)

- additionally given: nondegenerate tensor field $\tau \in \Gamma(T^{(0,2)}(T\mathcal{M}))$
- idea: **use additional structure** to find tangent projection

$$\begin{array}{ccc} X|_{\varphi(\check{m})} \in T_{\varphi(\check{m})}\mathcal{M} & \xrightarrow{\mathrm{d}\varphi|_{\varphi(\check{m})}} & T_{\check{m}}\check{\mathcal{M}} \\ \downarrow b_\tau & & \uparrow \sharp_\tau \\ T_{\varphi(\check{m})}^*\mathcal{M} & \xrightarrow{\mathrm{d}\varphi^*|_{\check{m}}} & T_{\check{m}}^*\check{\mathcal{M}} \end{array}$$

such that for $\check{x} \in \check{\mathcal{M}}$ and $v \in T_{\varphi(\check{x})}\mathcal{M}$

$$R_{\mathrm{GMG}}|_{\varphi(\check{x})}(v) := (\sharp_\tau \circ \mathrm{d}\varphi^*|_{\check{x}} \circ b_\tau)(v).$$

- GMG used for **Lagrangian systems (LMG)** and **Hamiltonian systems (SMG)**

Hamiltonian systems - Symplectic Manifold Galerkin (SMG)

Assume: constructed smooth embedding $\varphi \in \mathcal{C}^\infty(\check{\mathcal{M}}, \mathcal{M})$ fulfilling $\check{\omega} = \varphi^* \omega$ is nondegenerate.

Theorem (SMG-ROM [Buchfink, G. Haasdonk, Unger 2024])

The SMG-ROM is a Hamiltonian system $(\check{\mathcal{M}}, \check{\omega}, \check{\mathcal{H}})$ with reduced Hamiltonian $\check{\mathcal{H}} := \mathcal{H} \circ \varphi$.

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Assume: constructed smooth embedding $\varphi \in \mathcal{C}^\infty(\check{\mathcal{M}}, \mathcal{M})$ fulfilling $\check{\omega} = \varphi^* \omega$ is nondegenerate.

Theorem (SMG-ROM [Buchfink, G. Haasdonk, Unger 2024])

The SMG-ROM is a Hamiltonian system $(\check{\mathcal{M}}, \check{\omega}, \check{\mathcal{H}})$ with reduced Hamiltonian $\check{\mathcal{H}} := \mathcal{H} \circ \varphi$.

Lemma (Canonical Hamiltonian systems [Buchfink, G. Haasdonk, Unger 2024])

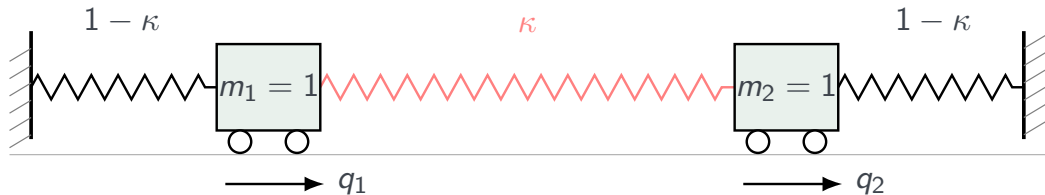
For $(\mathbb{R}^{2N}, \mathbb{J}_{2N}^T, \mathcal{H})$ and reduced symplectic manifold $(\check{\mathcal{M}}, \check{\omega}) = (\mathbb{R}^{2n}, \mathbb{J}_{2n}^T)$, it holds that

- ① *the SMG reduction evaluated at the base point $\varphi(\check{\mathbf{m}})$ equals the symplectic inverse*

$$\mathbf{R}|_{\varphi(\check{\mathbf{m}})}(\mathbf{v}) = \mathbf{D}\varphi|_{\check{\mathbf{m}}}^+ \mathbf{v} := \mathbb{J}_{2n} \mathbf{D}\varphi|_{\check{\mathbf{m}}}^T \mathbb{J}_{2N}^T \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbb{R}^{2N},$$

- ② *the SMG-ROM is consistent with [Buchfink, G., Haasdonk, 2023], and*
- ③ *if φ is linear, the SMG-ROM equals the symplectic Galerkin ROM introduced in [Peng, Mohseni, 2016], [Maboudi Afkham, Hesthaven 2017].*

Example: Three different reductions for coupled oscillators



Resulting in Hamiltonian System:

$$\underbrace{\frac{d}{dt} \begin{pmatrix} q_1(t) \\ q_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix}}_{\mathbf{x}(t)} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{=: \mathbb{J}_4} \nabla_{\mathbf{x}} \mathcal{H}_{\kappa} \begin{pmatrix} q_1(t) \\ q_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix}$$

where $0 < \kappa < 1$ and

$$\mathcal{H}_{\kappa}(\mathbf{x}(t)) = \frac{1}{2} (p_1(t)^2 + p_2(t)^2 + q_1(t)^2 + q_2(t)^2) - \kappa q_1(t) q_2(t).$$

Same trial manifold, different reductions

Choose $n = 1$ with embedding and point reductions

$$\varphi(q, p) = (q, q^2, p, 0), \quad \rho(\mathbf{x}) = (q_1, p_1), \quad \rho \circ \varphi = \text{id}_{\mathbb{R}^2}.$$

Further, we set

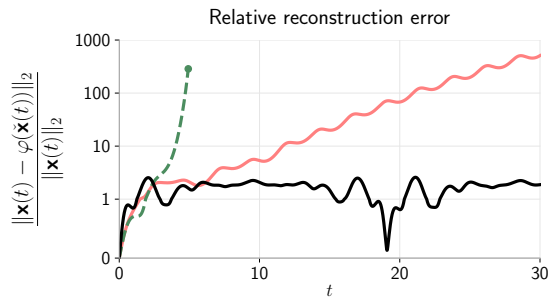
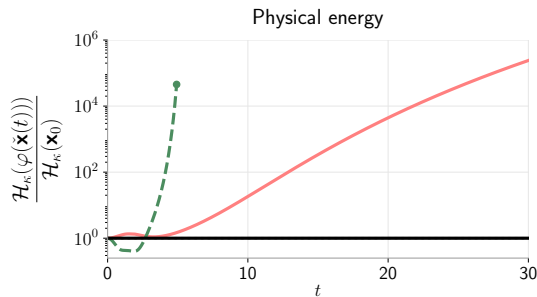
$$\mathbf{x}_0 = (-1, 1, 0, 0), \quad \text{thus } \check{\mathbf{x}}_0 = (-1, 0), \quad \kappa = \frac{3}{4}.$$

Leading to the following tangent reductions:

Method	τ	Tangent reduction R	Conservation of $\mathcal{H} \circ \varphi$
MPG	-	$D\rho$	no
MG	identity	$((D\varphi)^\top D\varphi)^{-1} (D\varphi)^\top$	no
SMG	symplectic form	$\mathbb{J}_2^\top (D\varphi)^\top \mathbb{J}_4$	yes

Long term behavior of solutions

Method	τ	Tangent reduction R	Conservation of $\mathcal{H} \circ \varphi$
MPG	-	$D\rho$	no
MG	identity	$((D\varphi)^\top D\varphi)^{-1} (D\varphi)^\top$	no
SMG	symplectic form	$\mathbb{J}_2^\top (D\varphi)^\top \mathbb{J}_4$	yes



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Snapshot-based generation of embedding and point reduction

- Consider training set of snapshots $S_{\text{train}} \subseteq S$ with
- $x_{\text{train}} \in S_{\text{train}}$ as *snapshots*.
- search embedding and the point reduction in given family of functions

$$\mathcal{F}_{\varphi, \varrho} := \{(\varphi, \varrho) \in \mathcal{C}^\infty(\check{\mathcal{M}}, \mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}, \check{\mathcal{M}}) \mid \varrho \text{ is a point reduction for } \varphi\}$$

- optimize over functional $\mathcal{L}: \mathcal{F}_{\varphi, \varrho} \rightarrow \mathbb{R}_{\geq 0}$ that measuring quality of approximation

$$(\varphi^*, \varrho^*) := \arg \min_{(\varphi, \varrho) \in \mathcal{F}_{\varphi, \varrho}} \mathcal{L}(\varphi, \varrho)$$

Use mean squared error (in coordinates)

$$\mathbf{L}(\varphi, \varrho) = \frac{1}{|S_{\text{train}}|} \sum_{x_{\text{train}} \in S_{\text{train}}} \|x_{\text{train}} - (\varphi \circ \varrho)(x_{\text{train}})\|^2.$$

Quadratic Manifolds

Embedding and point reduction are set to

$$\varphi_{\text{quad}}(\check{\mathbf{x}}) := \mathbf{A}_2 \check{\mathbf{x}}^{\otimes^2} + \mathbf{A}_1 \check{\mathbf{x}} + \mathbf{A}_0, \quad \varrho_{\text{quad}}(\mathbf{x}) := \mathbf{A}_1^T (\mathbf{x} - \mathbf{A}_0), \quad (2)$$

with

- $\mathbf{A}_2 \in \mathbb{R}^{N \times n(n+1)/2}$, $\mathbf{A}_1 \in \mathbb{R}^{N \times n}$, $\mathbf{A}_0 \in \mathbb{R}^N$,
- $(\cdot)^{\otimes^2}: \mathbb{R}^n \rightarrow \mathbb{R}^{n(n+1)/2}$, $\check{\mathbf{x}} \mapsto \check{\mathbf{x}}^{\otimes^2}$, denote the *symmetric Kronecker product*, which produces all pairwise products of components

Under the following conditions a point projection property can be shown

$$\mathbf{A}_1^T \mathbf{A}_1 = \mathbf{I}_n, \quad \mathbf{A}_1^T \mathbf{A}_2 = \mathbf{0}_{n \times \tilde{n}}$$

Known works include:

- Non-intrusive MOR (Geelen, Wright, Wilcox, 2022)
- Intrusive MOR (Barnett, Farhat, 2022)
- Hamiltonian Systems (Sharma, Mu, Buchfink, Geelen, G., Kramer, 2023)

Using Autoencoders: Point Projection Property

Consider ϱ as *encoder*, φ as the *decoder*, and the concatenation $\varphi \circ \varrho$ as an *autoencoder*

Theorem (Buchfink, G., Haasdonk, Unger, 2024)

Assume

- φ and ϱ are Lipschitz continuous with $C_\varphi, C_\varrho \geq 0$

Then, the point projection properties are fulfilled approximately s.t. for each $\check{\mathbf{x}} \in \mathbb{R}^n$

$$\begin{aligned} \|(\varrho \circ \varphi)(\check{\mathbf{x}}) - \check{\mathbf{x}}\| &\leq C_\varrho \sqrt{|S_{\text{train}}| L(\varphi, \varrho)} \\ &\quad + (C_\varrho C_\varphi + 1) \min_{\mathbf{x}_{\text{train}} \in S_{\text{train}}} \|\check{\mathbf{x}} - \varrho(\mathbf{x}_{\text{train}})\| \end{aligned}$$

Remark: the autoencoders in e.g., [Otto, Macchio, Rowley \(2023\)](#), [Botteghi, G., Brune \(2026\)](#) fulfill the projection property exactly.

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Example: Structure-preserving MOR - Wave Equation

Linear 1D Wave Equation: $\Omega := [-0.5, 0.5]$, $t \in [0, 1]$, $\mu = 0.5$

$$\begin{aligned}\partial_{tt}^2 \varphi(t, \xi; \mu) &= \mu^2 \partial_{\xi\xi}^2 \varphi(t, \xi; \mu), & \text{on } I \times \Omega, \\ \varphi(0, \xi; \mu) &= \varphi_0(\xi; \mu), & \forall \xi \in \Omega, \\ \partial_t \varphi(0, \xi; \mu) &= -\mu \partial_\xi \varphi_0(\xi; \mu), & \forall \xi \in \Omega, \\ \varphi(t, \xi; \mu) &= 0, & \forall t \in I, \xi \in \{-0.5, 0.5\},\end{aligned}$$

Discrete FOM model:

$$\dot{\mathbf{y}}(t) = \begin{pmatrix} \dot{\mathbf{q}}(t) \\ \dot{\mathbf{p}}(t) \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix}, \quad \text{with } \mathbf{A} := \begin{pmatrix} \mathbf{0} & \mathbb{I}_n \\ \frac{1}{4} \mathbf{D}_{xx} & \mathbf{0} \end{pmatrix}$$

- Time discretization: **implicit midpoint** with $n_t = 4000$,
- State discretization: $2n = 4096$.

Weakly Symplectic Deep Convolutional Autoencoder

Loss function:

$$\mathcal{L}(\theta) := \alpha \mathcal{L}_{\text{data}}(\theta) + (1 - \alpha) \mathcal{L}_{\text{symp}}(\theta),$$

with data and symplecticity loss

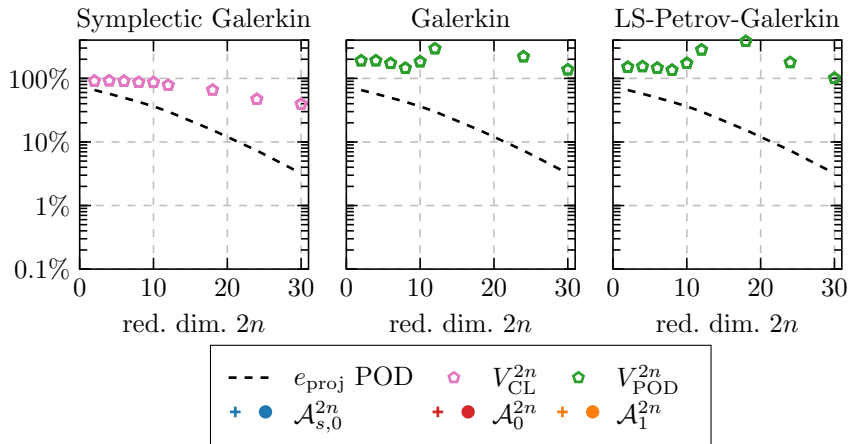
$$\mathcal{L}_{\text{data}}(\theta) := \frac{1}{N|\mathcal{X}|} \sum_{\mathbf{x} \in \mathcal{X}} \|\mathbf{x} - \varphi(\varrho(\mathbf{x}; \theta); \theta)\|^2,$$

$$\mathcal{L}_{\text{symp}}(\theta) := \frac{1}{(2n)^2 |\mathcal{X}|} \|(\mathbf{D}_{\check{\mathbf{x}}} \varphi(\cdot; \theta)|_{\varrho(\mathbf{x}; \theta)})^T \mathbb{J}_{2N} \mathbf{D}_{\check{\mathbf{x}}} \varphi(\cdot; \theta)|_{\varrho(\mathbf{x}; \theta)} - \mathbb{J}_{2n}\|_{\mathbb{F}^2},$$

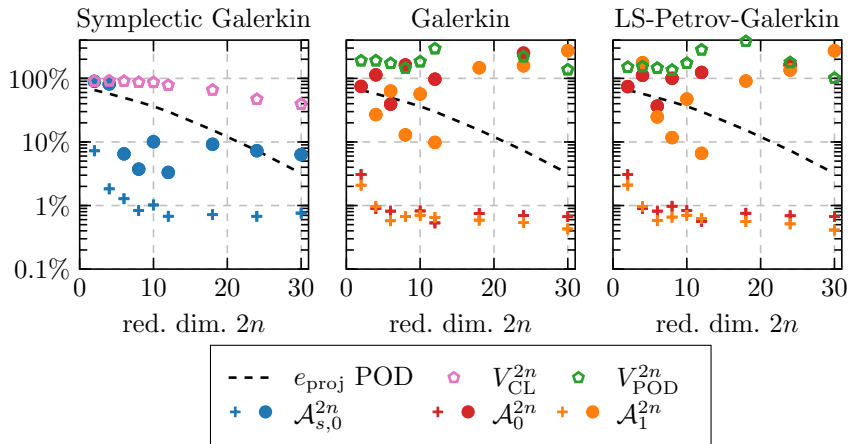
Autoencoder architectures: basis sizes $2n \in \{2, 4, \dots, 12, 18, 24, 30\}$

- weakly symplectic DCA $\mathcal{A}_{s,0}^{2n}$ architecture,
- a non-symplectic copy $\mathcal{A}_0^{2n}$,
- completely different non-symplectic DCA architecture $\mathcal{A}_1^{2n}$.

Autoencoder Numerical Results: Projection Error/Reduction Error

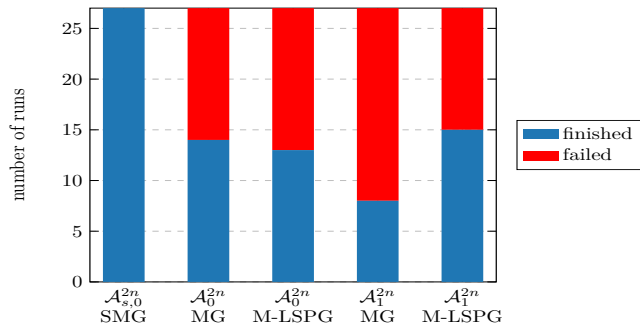


Autoencoder Numerical Results: Projection Error/Reduction Error



Autoencoder Numerical Results: Reduced Solution Runs

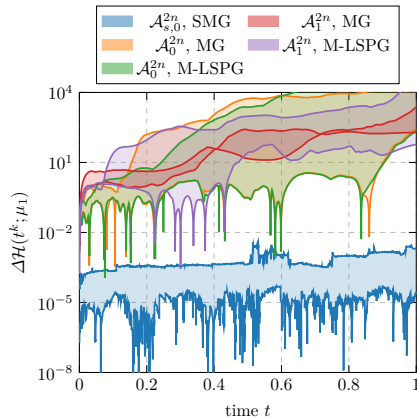
- Number of **successful** and **failed** reduced simulation runs,
- 1 weakly symplectic (left) and 4 general autoencoders
- for 3 (in training unseen) parameters and 9 reduced dimensions $2n \in \{2, 4, \dots, 12, 18, 24, 30\}$.



Autoencoder Numerical Results: Hamiltonian

Error in Hamiltonian:

$$\Delta\mathcal{H}(t; \mu) = |\mathcal{H}(\mathbf{x}(t; \mu); \mu) - \mathcal{H}(\tilde{\mathbf{x}}(t; \mu); \mu)|$$



From weak to strong: Manifold Cotangent Lift^[1]

Split the procedure of constructing a symplectic map for $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ into two parts:

- 1 Fit nonlinear approximation for $\mathbf{q}$ only

$$\varphi_{\mathbf{q}} : \mathbb{R}^n \rightarrow \mathbb{R}^N.$$

- 2 Construct nonlinear approximation for $\mathbf{p}$ based on step 1 with $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2N}$

$$\varphi(\mathbf{q}_r, \mathbf{p}_r) := \begin{pmatrix} \varphi_{\mathbf{q}}(\mathbf{q}_r) \\ \varphi_{\mathbf{p}}(\mathbf{q}_r, \mathbf{p}_r) \end{pmatrix} := \begin{pmatrix} \varphi_{\mathbf{q}}(\mathbf{q}_r) \\ \mathbf{p}_{\text{ref}} + ((D_{\mathbf{q}}\varphi_{\mathbf{q}}(\mathbf{q}_r)^{\dagger}))^T \mathbf{p}_r \end{pmatrix} \quad (5)$$

where $D_{\mathbf{q}}\varphi_{\mathbf{q}}(\mathbf{q}_r)^{\dagger}$ is the Moore-Penrose inverse of Jacobian $D_{\mathbf{q}}\varphi_{\mathbf{q}}(\mathbf{q}_r)$.

Theorem (Sharma, Mu, Buchfink, Geelen, G., Kramer, 2023)

If $\varphi_{\mathbf{q}} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^N)$ is an immersion, then the mapping in (5) is a symplectic map.

^[1]H. Sharma, H. Mu, P. Buchfink, R. Geelen, SG., B. Kramer. *Symplectic model reduction of Hamiltonian systems using data-driven quadratic manifolds*, Computer Methods in Applied Mechanics and Engineering, 2023.

Example: Nonlinear Parametric Wave Equation

Nonlinear 2D Wave Equation: $\Omega := (-10, 10) \times (-10, 10)$, $t \in (0, 8]$, $\mu \in [0.1, 1]$

$$\begin{aligned}\frac{\partial^2 \varphi}{\partial t^2}(x, y, t; \mu) &= \frac{\partial^2 \varphi}{\partial x^2}(x, y, t; \mu) + \frac{\partial^2 \varphi}{\partial y^2}(x, y, t; \mu) - \mu \varphi(x, y, t; \mu)^3, & \text{on } I \times \Omega, \\ \varphi(0, \xi; \mu) &= 2\text{sech}(\cosh(x^2 + y^2)), & \forall x, y \in \Omega, \\ \partial_t \varphi(0, \xi; \mu) &= 0, & \forall x, y \in \Omega, \\ \varphi(-10, y, t; \mu) &= \varphi(10, y, t; \mu), \varphi(x, -10, t; \mu) = \varphi(x, 10, t; \mu) & \forall t \in I.\end{aligned}$$

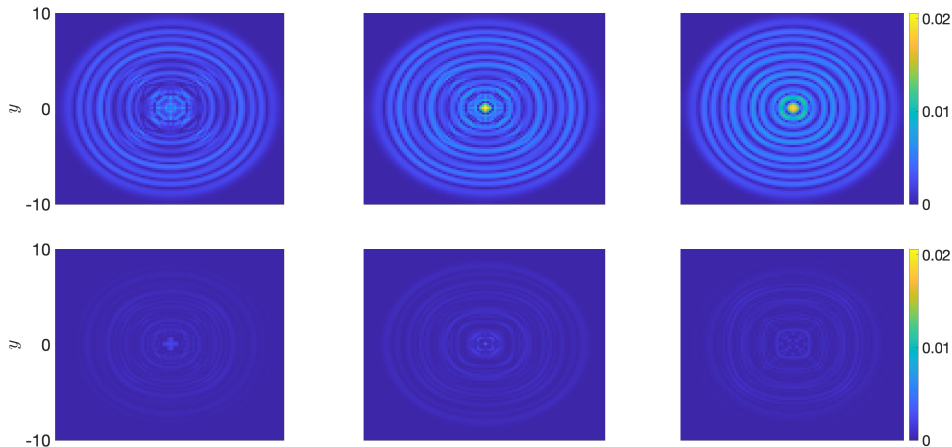
(Discrete) Hamiltonian FOM:

$$\begin{pmatrix} \dot{\mathbf{q}}(t; \mu) \\ \dot{\mathbf{p}}(t; \mu) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I_n \\ \mathbf{D}_{\text{fd}, 2d} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}(t; \mu) \\ \mathbf{p}(t; \mu) \end{pmatrix} - \mu \begin{pmatrix} \mathbf{0} \\ \mathbf{q}(t; \mu)^3 \end{pmatrix}$$

- Time discretization: **implicit midpoint** with $\Delta t = 0.1$
- Training data: $|\mathcal{P}_{\text{train}}| = 10$ equidistantly distributed in $\mathcal{P}$

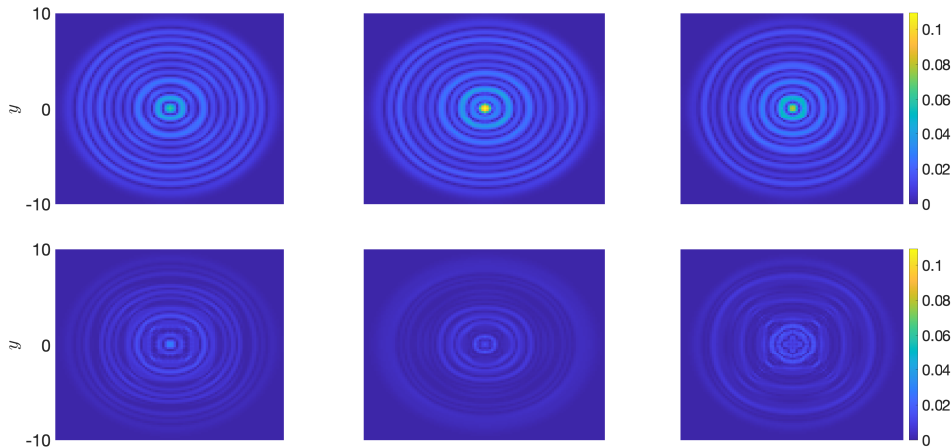
Parameter Extrapolation Study ($\mu = 1.25, 2r = 48, t = 6, 7, 8$)

- Upper row: State Error; linear symplectic ROM; lower row: SMG-QMCL-ROM



Parameter Extrapolation Study ($\mu = 3, 2r = 48, t = 6, 7, 8$)

- Upper row: State Error; linear symplectic ROM; lower row: SMG-QMCL-ROM



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Conclusion & Future work

Conclusion:

- Presented nonlinear and structure-preserving MOR methods on manifolds^[2]
- Symplectic Manifold Galerkin preserves Hamiltonian system within ROMs
- Nonlinear embedding and reduction maps realized (here) with
 - weakly symplectic autoencoder
 - strongly symplectic quadratic embedding via manifold cotangent lift

Future Work:

- look into different structures, e.g., port-Hamiltonian, metriplectic
- Embed reduced order model into control loop

^[2]Under the hood: geometric formulation: P. Buchfink, SG., B. Haasdonk, B. Unger. *Model Reduction on Manifolds: a differential geometric framework*, Physica D: Nonlinear Phenomena, 2024.