

Adaptive methods for nonlinear, doubly-degenerate diffusion equations

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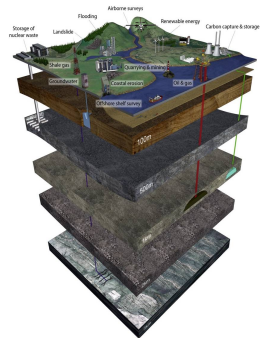
Motivation

Societal relevance

- ▶ Groundwater management;
- ▶ Geological CO₂ storage;
- ▶ Biofilm growth in porous media;
- ▶ Oil recovery;
- ▶ Phase change (Stefan).

Mathematical challenges

- ▶ Nonlinear, possibly degenerate equations (slow/fast diffusion), multi-valued operators;
- ▶ Heterogeneous media (jump-type discontinuities), interfaces.



British Geological Survey
[https://www.bgs.ac.uk/
research/images/
manageTheSubsurface.jpg](https://www.bgs.ac.uk/research/images/manageTheSubsurface.jpg)

Outline

- ▶ Motivation
- ▶ Application: variably saturated flow in porous media
- ▶ Mathematical framework
- ▶ Linear iterative schemes
- ▶ Space-time discretisation (adaptivity & linearisation)

The Richards equation (porous media flows)

$$\partial_t \theta(\psi) - \nabla \cdot (k_r(\theta(\psi)) \nabla (\psi + z)) = 0$$

Features:

- ▶ Given functions relationship as $\theta = \theta(\psi)$, $k_r = k_r(\theta(\psi))$
- ▶ Doubly degenerate problem: $\theta' = 0$ (fast diffusion), resp. $\theta' \rightarrow \infty$, or $k_r = 0$ (slow diffusion)
- ▶ Change of type, free boundaries, lack in regularity, infinite gradients

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- ▶ Change of type, free boundaries, lack in regularity, infinite gradients
- ▶ Kirchhoff transform:

$$w = \beta(\psi) = \int_0^\psi k_r(\theta(z)) dz, \quad \text{thus} \quad \nabla w = k_r(\theta(\psi)) \nabla \psi,$$

$$\partial_t u = \Delta w + \partial_z (k_r(u)) \quad \text{with} \quad u = \theta(\beta^{-1}(w)).$$

Note: $w = \beta(\theta^{-1}(u))$; w is more regular, but $0 \leq \frac{\theta'(\cdot)}{\beta'(\cdot)} \leq \infty$

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Mathematical framework

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}_0$) be a bounded, Lipschitz domain. Consider

$$\partial_t u + \nabla \cdot \mathbf{F}(u) = \Delta w, \text{ in } (0, T] \times \Omega, \quad \text{with } w \in \Phi(u),$$

and for given boundary (e.g. homogeneous Dirichlet) and initial conditions.

Assumptions:

(A.1) $\Phi : [0, b) \rightarrow [0, \infty)$ is C^1 , increasing and locally Lipschitz;
 Φ can become multivalued, or infinite at $b \in (0, \infty]$;
 Φ' is strictly increasing in $[0, \xi_0)$ for some $\xi_0 \in (0, 1]$.

(A.2) $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^n$ is s.t. there exists an $L_f > 0$ s.t.

$$|\mathbf{F}(u) - \mathbf{F}(v)|^2 \leq L_f |u - v| |\Phi(u) - \Phi(v)|.$$

Richards equation (example): $\Phi(u) = \beta(\theta^{-1}(u))$.

Examples

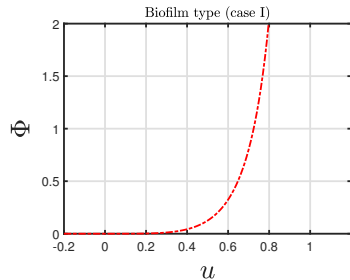
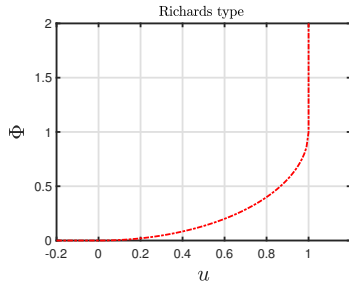


Figure: Φ as a multivalued graph (left); Φ may become infinite (right).

Idea

Use the auxiliary functions:

$$b, B : \mathbb{R} \rightarrow \mathbb{R} \quad \text{s.t.} \quad B \circ b^{-1} = \Phi, \quad 0 \leq b', B' \leq 1, \quad \text{and} \quad 1 \leq b' + B'.$$

and an additional unknown s to reformulate the equation as¹

$$\partial_t u + \nabla \cdot \mathbf{F}(u) = \Delta w, \quad \text{in } (0, T] \times \Omega,$$

$$\text{with } u = b(s) \quad \text{and} \quad w = B(s), \quad \text{a.e. in } (0, T] \times \Omega.$$

¹Brenner, Cancès '17

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Construction of b and B :

$$b(s) = \int_0^s \min \left\{ 1, \frac{1}{\Phi'(\rho)} \right\} d\rho, \quad B(s) = \int_0^s \min \{ 1, \Phi'(\rho) \} d\rho.$$

¹Brenner, Cancès '17

The functions Φ , b and B

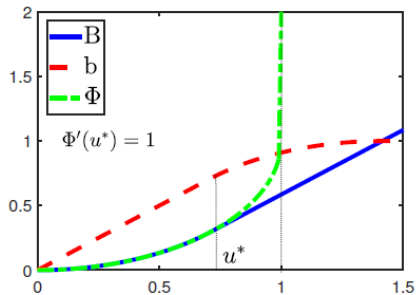


Figure: The function Φ and the auxiliary functions b , B .

Weak solutions

Problem WP: (homogeneous Dirichlet boundary conditions)

Find $(u, w) \in H^1(0, T; H^{-1}(\Omega)) \times L^2(0, T; H_0^1(\Omega))$ s.t. $w \in \Phi(u)$ a.e., $u(0) = u_0$ (in H^{-1}) and for all $\varphi \in L^2(0, T; H_0^1(\Omega))$,

$$(\partial_t u, \varphi) + (\nabla w - \mathbf{F}(u), \nabla \varphi) = 0.$$

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- ▶ Existence, uniqueness of weak solutions, continuity: van Duijn, Peletier '82, Alt, Luckhaus '83, Otto '96, Carillo '99, P, Schweizer '11, Ziemer '82, Cancès, Gallouët '11, Horger, Kuttler, Wohmuth, Zhigun '14, Liao '20, Mitra, Sonner '23
- ▶ Often: $\Phi(\cdot) \approx \Phi_\epsilon(\cdot)$, with $0 < \epsilon \leq \Phi'_\epsilon(\cdot) \leq \frac{1}{\epsilon}$ (regularization)

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- ▶ Often: $\Phi(\cdot) \approx \Phi_\epsilon(\cdot)$, with $0 < \epsilon \leq \Phi'_\epsilon(\cdot) \leq \frac{1}{\epsilon}$ (regularization)
- ▶ Numerical schemes, Euler implicit + FEM/MFEM, FV, MPFA, DG, GDM: Nochetto, Verdi '88, Jäger, Kačur '91, '95, Rulla, Walkington '96, Arbogast, Wheeler, Zhang '96, Yotov '97, Riviere, Wheeler, Banas '00, '02, Radu, P., Knabner '04, '09, Wheeler, Yotov '05, Eymard, Vohralik, Hilhorst '06, Klausen, Radu '08, Bause '08, Peszynska, Yi '08, Vohralik, Wheeler '14, Cancès, P., Vohralik '14, Berardi et al. '20, Droniou, Le '20, Cancès et al. '21, Berardi, Difonzo '22

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Euler implicit discretisation

Scalar equation

$$\partial_t b(s) + \nabla \cdot (\mathbf{F}(b(s)) - \nabla B(s)) = 0,$$

¹Cancés et al. '21, Smeets et al. '25, Javed, Mitra, P. '26

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Double split

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Regularization: $\varepsilon \geq 0$

$$b_\varepsilon(s) = \int_0^s \max\{\varepsilon, b'(z)\} dz \text{ and } B_\varepsilon(s) = \int_0^s \max\{\varepsilon, B'(z)\} dz$$

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Euler implicit¹ $n \in \mathbb{N}$, $\tau = T/n$, $t_k = k\tau$, $k = \overline{1, n}$, $\zeta^k \approx \zeta(t_k)$, $\zeta \in \{s, u, w\}$

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$$b_\varepsilon(s^k) - b_\varepsilon(s^{k-1}) + \tau \nabla \cdot (\mathbf{F}(b_\varepsilon(s^k)) - \nabla B_\varepsilon(s^k)) = 0, \text{ or}$$

$$u^k - u^{k-1} + \tau \nabla \cdot \mathbf{F}(u^k) = \tau \Delta w^k, \quad u^k = b_\varepsilon(s^k), \quad w^k = B_\varepsilon(s^k),$$

+ your favorite spatial discretisation (FV, MPFA, (M)FEM, DG, GDM, ...)

¹Cancés et al. '21, Smeets et al. '25, Javed, Mitra, P. '26

Iterative schemes: Newton

Iteration index i : assume $s^{k,i-1}$ and s^{k-1} given. Solve¹

$$\begin{aligned} & b'_\varepsilon(s^{k,i-1})(s^{k,i} - s^{k,i-1}) + b_\varepsilon(s^{k,i-1}) - b_\varepsilon(s^{k-1}) \\ & + \tau \nabla \cdot \left((\mathbf{F} \circ b_\varepsilon)'(s^{k,i-1})(s^{k,i} - s^{k,i-1}) + \mathbf{F}(b_\varepsilon(s^{k,i-1})) \right) \\ & = \tau \Delta \left(B_\varepsilon(s^{k,i-1}) + B'_\varepsilon(s^{k,i-1})(s^{k,i} - s^{k,i-1}) \right). \end{aligned}$$

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²Radu, P, Knabner '05

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Rem: Quadratic convergence is guaranteed (mathematically rigorous) if²:

1. b', B' are Lipschitz, and the model is regularised (i.e. $\varepsilon > 0$),
2. the initial guess is $s^{k,0} = s^{k-1}$, and
3. the time step is subject to severe constraints, $\tau = O(h^d)$ (h - mesh size).

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Improvements: Convex-concave splitting, change of parameterisation³

¹Celia et al. '90, Bergamaschi, Putti '99

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Iterative schemes: L-scheme

With $L > 0$ large enough, assume $s^{k,i-1}$ and s^{k-1} given. Solve¹

$$\begin{aligned} L(s^{k,i} - s^{k,i-1}) + b(s^{k,i-1}) - b(s^{k-1}) + \tau \nabla \cdot \mathbf{F}(b(s^{k,i-1})) \\ = \tau \Delta \left(B(s^{k,i-1}) + L(s^{k,i} - s^{k,i-1}) \right). \end{aligned}$$

¹P, Radu, Knabner '04, List, Radu '16, Radu et al. '18, Cancès et al '21, Smeets et al. '25, Javed, Mitra, P. '26

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Rem: Convergence is linear/contraction, but under less restrictive conditions!

1. b, B are Lipschitz and increasing, $L \geq \frac{1}{2} \max\{L_b, L_B\}$ (here $L \geq \frac{1}{2}$);
2. Mild constraints on τ ($\tau \leq C$), not depending on h or the discretisation;
3. No regularisation needed, $\varepsilon = 0$;
4. No restriction for the initial guess;
5. Convergence rate $\rho = \|s^{k,i} - s^k\| / \|s^{k,i-1} - s^k\|$ decreases with L ,
 $\rho = L / (L + C\tau)$;
6. Works for Hölder-continuous b, B (e.g. $b(s) = Cs^\alpha$ for some $\alpha \in (0, 1)$),
worse convergence rate.

¹P, Radu, Knabner '04, List, Radu '16, Radu et al. '18, Cancès et al '21, Smeets et al. '25, Javed, Mitra, P. '26

Iterative schemes: M(odified L)-scheme

Assume that $\Lambda > 0$ exists s.t. $\|s^k - s^{k-1}\|_{L^\infty(\Omega)} \leq \tau \Lambda$, and b', B' - Lipschitz.
Take $s^{k,0} = s^{k-1}$. For $i > 0$, given $s^{k,i-1}$ and s^{k-1} , solve¹

$$\begin{aligned} L_{b,k}^i(s^{k,i} - s^{k,i-1}) + b_\varepsilon(s^{k,i-1}) - b_\varepsilon(s^{k-1}) + \tau \nabla \cdot \mathbf{F}(b(s^{k,i-1})) \\ = \tau \Delta \left(B_\varepsilon(s^{k,i-1}) + L_{B,k}^i(s^{k,i} - s^{k,i-1}) \right), \text{ where} \end{aligned}$$

$L_{b,k}^i = \mathcal{M}_b \tau + \max\{b'_\varepsilon(s^{k,i-1}), \mathcal{M}_b \tau\}$ with $\mathcal{M}_b \geq \Lambda \|b''\|_\infty$ and similarly for $L_{B,k}^i$.

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Lemma: Assume $b \in C^2$, the initial data in H^1 , and that $B' \geq m_B > 0$ (but still, $\varepsilon \geq 0$). One has $\|s^{k,i} - s^k\|_{L^\infty(\Omega)} \leq \tau\Lambda$ for all i .

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Theorem: The modified L -scheme converges linearly in H^1 with convergence rate

$$\rho = \sqrt{\mathcal{M}/(\mathcal{M} + C_1)},$$

for some $C_1 > 0$. In the non-degenerate case (or if $\varepsilon > 0$), a $C_2 > 0$ exists s.t.

$$\rho = \min\{\rho, C_2\tau\}.$$

¹Mitra, P. '19, Javed, Mitra, P. '26

Double split formulation

Split formulation¹ (three unknowns, s, u, w !)

$$\begin{aligned}\partial_t u + \nabla \cdot \mathbf{F}(u) &= \Delta w, \\ u &= b(s), \\ w &= B(s)\end{aligned}$$

and its Euler implicit discretisation (no regularization needed!)

$$\begin{aligned}u^k - u^{k-1} + \tau \nabla \cdot \mathbf{F}(u^k) &= \tau \Delta w^k, \\ u^k &= b(s^k), \\ w^k &= B(s^k)\end{aligned}$$

(+ your favourite spatial discretisation scheme).

¹Cancès et al '21, Smeets et al. '25, Javed, Mitra, P. '26

Double split formulation: iterative schemes

For $i = 1, 2, \dots$, given $s^{k,i-1}, u^{k,i-1}$, solve

$$u^{k,i} - \tau \Delta w^{k,i} = u^{k-1} + \tau \nabla \cdot \mathbf{F}(u^{k,i-1})$$

$$u^{k,i} - L_b^{k,i} s^{k,i} = b(s^{k,i-1}) - L_b^{k,i} s^{k,i-1},$$

$$w^{k,i} - L_B^{k,i} s^{k,i} = B(s^{k,i-1}) - L_B^{k,i} s^{k,i-1}.$$

Start with $s^{k,0} = s^{k-1}$.

Double split formulation: iterative schemes

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Start with $s^{k,0} = s^{k-1}$. The choice of $L_b^{k,i}, L_B^{k,i}$ defines the scheme.

- ▶ Newton: $L_b^{k,i} = b'(s^{k,i-1}), L_B^{k,i} = B'(s^{k,i-1})$
- ▶ L -scheme ($0 < \epsilon \ll 1$): $L_b^{k,i} = L_B^{k,i} = 1 + \epsilon \ (\geq \max\{\|b'\|_\infty, \|B'\|_\infty\})$;
- ▶ M -scheme:

$$\begin{aligned} L_b^{k,i} &= \min \left\{ 1 + \epsilon, \mathcal{M}\tau + \max \{ \mathcal{M}\tau, b'(s^{k,i-1}) \} \right\}, \\ L_B^{k,i} &= \min \left\{ 1 + \epsilon, \mathcal{M}\tau + \max \{ \mathcal{M}\tau, B'(s^{k,i-1}) \} \right\}, \end{aligned}$$

where $\mathcal{M} \geq \Lambda \max\{\|b''\|_\infty, \|B''\|_\infty\}$, $\Lambda > 0$ s.t. $\|s^k - s^{k-1}\|_{L^\infty(\Omega)} \leq \tau \Lambda$.

Double split formulation: iterative schemes

Theorem (convergence)¹: Under the assumptions for the L/M -scheme, one has

$$\lim_{i \rightarrow \infty} \left[\|\nabla(w^{k,i} - B(s^k))\| + \|s^{k,i} - s^k\|_{L^1} + \|u^{k,i} - b(s^k)\|_{L^1} \right] = 0$$

Rem: Better convergence in the non-degenerate case (contraction).

Stopping criterion:

$$\left\| (L_b^{k,i})^{1/2} (L_B^{k,i})^{-1/2} (s^{k,i} - s^{k,i-1}) \right\|_{L^2(\Omega)} \leq 10^{-5}$$

¹Javed, Mitra, P. '26

Double split formulation: adaptive choice of $\mathcal{M}$

A-posteriori analysis (Mitra, Vohralik '23, Stokke et al '23):

Let $V = L^2(\Omega) \times H_0^1(\Omega)$ and define the residual $\mathcal{R}_k^i : V \rightarrow V^*$,

$$\begin{aligned} \langle \mathcal{R}_k^i((s, w)), (\psi, \varphi) \rangle &= (b(s) - u_{k-1}, \varphi) + \tau(\nabla w, \nabla \varphi) \\ &\quad - \tau(\mathbf{F}(b(s)), \nabla \varphi) + (L_b^{k,i}(B(s) - w), \psi). \end{aligned}$$

Define the *iteration-dependent* bilinear form $a_k^i : V \times V \rightarrow \mathbb{R}$,

$$a_k^i((s, w), (\xi, \phi)) := (L_b^{k,i}s, \phi) + \tau(\nabla w, \nabla \phi) + (L_b^{k,i}(L_B^{k,i}s - w), \xi).$$

Note: One has $a_k^i((s, w), (s, w)) = \int_{\Omega} L_b^{k,i} L_B^{k,i} s^2 + \tau |\nabla w|^2 dx$. This gives the *iteration-dependent norms* on V , respectively V^* ,

$$\|(s, w)\|_{1,i} := [a_k^i((s, w), (s, w))]^{\frac{1}{2}}, \quad \text{for any } (s, w) \in V,$$

$$\|\ell\|_{-1,i} := \sup_{(\psi, \phi) \in V} \frac{\ell((\psi, \phi))}{\|(\psi, \phi)\|_{1,i}}, \quad \text{for any } \ell \in V^*.$$

Double split formulation: adaptive choice of $\mathcal{M}$

Given $i > 0$ and $s_k^0 = s_{k-1}, s_k^{i-1} \in L^2(\Omega)$, at iteration i one finds $(s_k^i, w_k^i) \in V$ s.t.

$$a_k^i((s^{k,i} - s^{k,i-1}, w^{k,i} - w^{k,i-1}), (\psi, \phi)) = -\langle \mathcal{R}_k^i((s^{k,i-1}, w^{k,i-1})), (\psi, \phi) \rangle_{V^* \times V}.$$

for all $(\psi, \phi) \in V$. With this, update

$$u_n^{k,i} = b(s^{k,i-1}) + L_b^{k,i}(s^{k,i} - s^{k,i-1}).$$

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Define the *linearisation error*

$$\mathcal{E}^i = \|(s^{k,i} - s^{k,i-1}, w^{k,i} - w^{k,i-1})\|_{1,i},$$

and, with $\gamma^{k,i} := (L_b^{k,i} / L_B^{k,i})^{1/2}$, the indicators

$$\eta_{\pm}^i := \left\| \gamma^{k,i} (w^{k,i-1} - B(s^{k,i-1})) \pm (\gamma^{k,i})^{-1} (u^{k,i-1} - b(s^{k,i-1})) \right\|.$$

Theorem (a-posteriori estimate). Let $\mathcal{L}_{a,k}^i > 0$ give the boundedness of a_k^i , then

$$\mathcal{E}^i \leq \|\mathcal{R}_k^i((s^{k,i-1}, w^{k,i-1}))\|_{-1,i} \leq \mathcal{L}_{a,k}^i \mathcal{E}^i, \text{ and } \frac{1}{2}(\eta_+^i - \eta_-^i) \leq \mathcal{E}^i \leq \frac{1}{2}(\eta_+^i + \eta_-^i).$$

Double split formulation: adaptive choice of $\mathcal{M}$

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Rem: Fully computable; choose $\mathcal{M}$ minimizing $\eta_+^i + \eta_-^i$ ("adaptive").

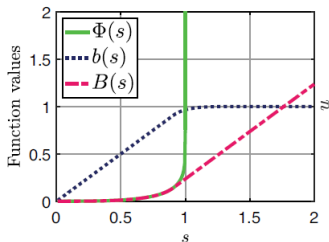
Numerical example: Richards equation

Domain and coefficient functions:

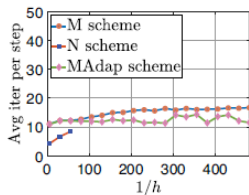
1. $\Omega = (-8, 8)^d$ ($d = 1$ or 2)
2. With $m = 0.8$, $n = \frac{1}{1-m}$,

$$k_r(\theta) = \theta^{1/2} (1 - (1 - \theta^{1/m})^m)^2, \text{ and } \theta(\psi) = \begin{cases} (1 + (-\psi)^n)^{-m}, & \psi < 0, \\ 1, & \psi \geq 0. \end{cases}$$

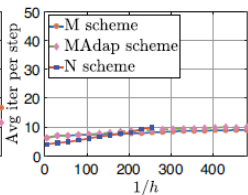
The mapping Φ and the functions b and B are determined by quadrature



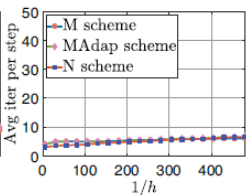
Richards: average number of iterations



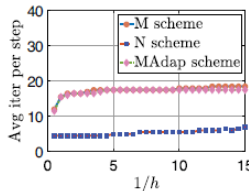
(A) $\tau = 10^{-1}$



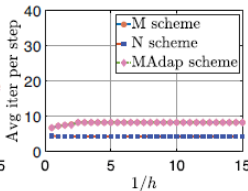
(B) $\tau = 10^{-1.5}$



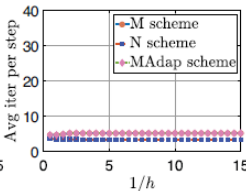
(C) $\tau = 10^{-2}$



(D) $\tau = 10^{-1}$



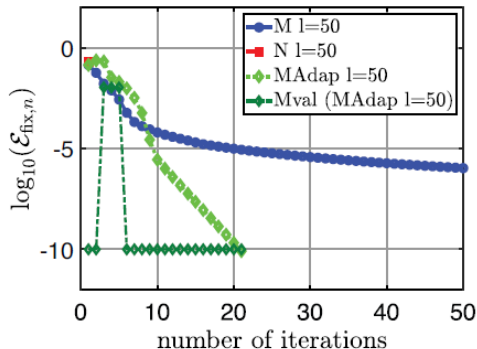
(E) $\tau = 10^{-1.5}$



(F) $\tau = 10^{-2}$

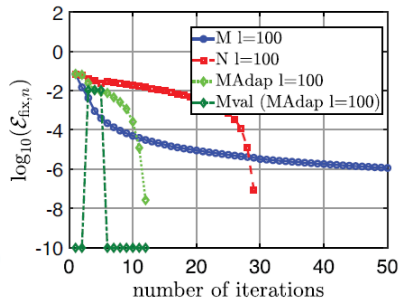
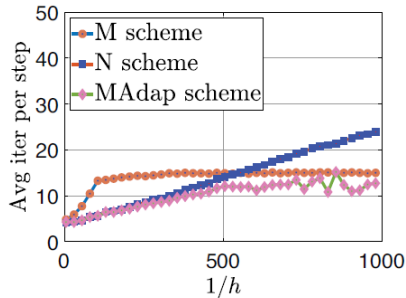
Average number of iterations per time step, different τ and h , $d = 1$ (upper row) and $d = 2$ (lower row)

Richards with gravity: convergence behaviour



Convergence behaviour for $\tau = 0.1$ and $h = 0.002$ ($d = 1$), case with convection:
Newton diverges, M slows down, adaptive M - rapid decay (see the values of M)

Richards, no gravity: convergence behaviour



Convergence behaviour, $\tau = 0.1$ ($d = 1$): different h values increasing number of iterations (left); number of iterations for $h = 0.001$ (right)

Outline

- ▶ Motivation
- ▶ Application: variably saturated flow in porous media
- ▶ Mathematical framework
- ▶ Linear iterative schemes
- ▶ Space-time discretisation (adaptivity & linearisation)

Space-time approach

Doubly degenerate equation in
the space-time domain

$$Q = (0, T) \times \Omega$$

$$\partial_t b(s) = \Delta B(s) + f \text{ in } Q,$$

with initial and boundary
conditions

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Sharp gradients $\Rightarrow$ small $\Delta t, h$;
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Idea:

adaptive space - time elements
linear iterations
decide: iteration vs. refinement

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The space-time L-scheme

Time derivative

$$b(s) \longrightarrow b(s^i) + L(s^{i+1} - s^i)$$

Diffusion term

$$B(s) \longrightarrow B(s^i) + L(s^{i+1} - s^i)$$

Idea: Linearize $b(s)$ and $B(s)$ around s^i with a stabilization parameter $L \geq 1$, ensuring robustness and convergence.

²Javed, Mitra, P. '26

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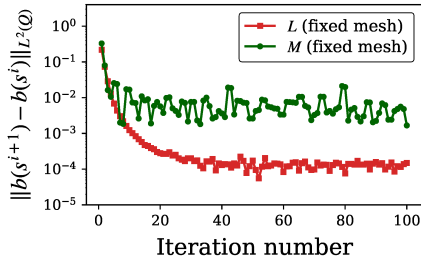
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Thm²: $b(s^i) \rightarrow b(s)$ in $L^2(0, T; H^{-1})$, $\int_0^t B(s^i) \rightarrow \int_0^t B(s)$ in $L^\infty(0, T; L^2)$

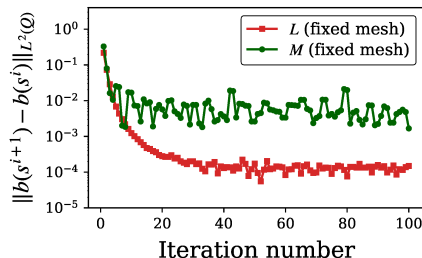
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Convergence behaviour (fixed mesh)



Iteration error, L -scheme vs. (regularised) Newton: note the stagnation

Convergence behaviour (fixed mesh)

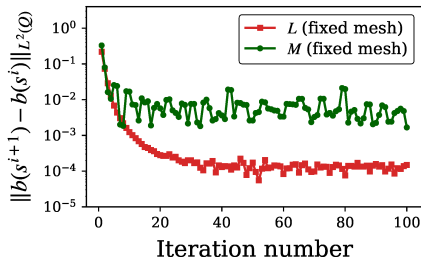


Iteration error, L -scheme vs. (regularised) Newton: note the stagnation

Idea: estimate $[\mathcal{E}(s^i)]^2 = L^2 \|s^{i+1} - s^i\|_{\mathcal{V}}^2$ by the error components

$$\mathcal{E}_{\text{lin}}^i := \|s_{h\tau}^{i+1} - s^i\|_{\mathcal{V}}, \mathcal{E}_{\text{disc}}^H(s_{h\tau}^{i+1}) := \left(\left\| \mathcal{R}_{\text{lin}}(s_{h\tau}^{i+1}, s^i) \right\|_{L^2(0, T; H^{-1}(\Omega))}^2 + L^2 \|s_{h\tau}^{i+1}(0) - s_0\|^2 \right)^{\frac{1}{2}}$$

Convergence behaviour (fixed mesh)



Iteration error, L -scheme vs. (regularised) Newton: note the stagnation

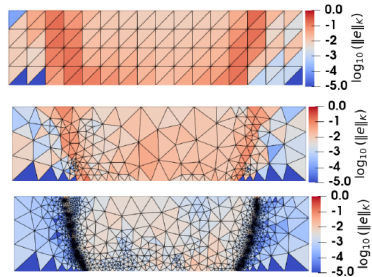
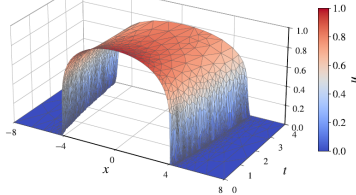
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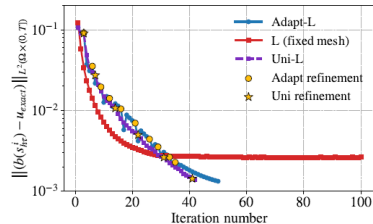
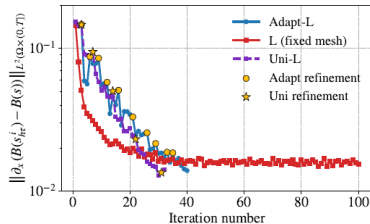
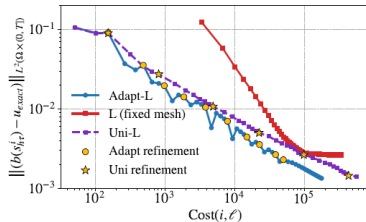
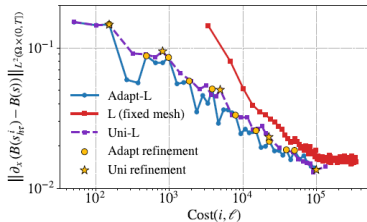
$$\mathcal{E}(s^i) \leq \mathcal{E}_{\text{disc}}^H + L \mathcal{E}_{\text{lin}}^i \quad \textbf{(Reliability)} \quad \text{—} \quad \textit{no error goes undetected}$$

$$\mathcal{E}_{\text{disc}}^H \leq \mathcal{E}(s^i) + L \mathcal{E}_{\text{lin}}^i \quad \textbf{(Efficiency)} \quad \text{—} \quad \textit{no unnecessary refinement}$$

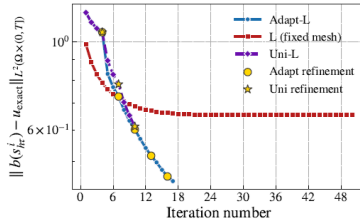
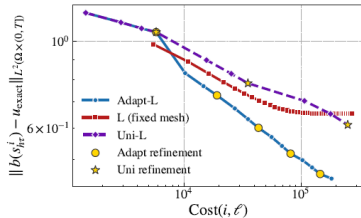
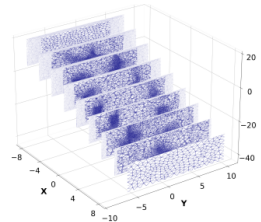
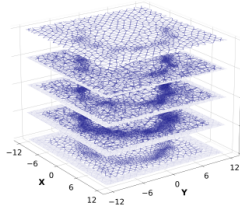
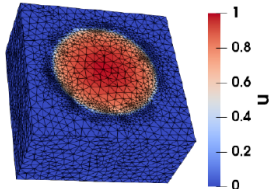
Porous medium equation (1D): space-time mesh



PME(1D): computational cost and error decay



PME(2D): mesh, cost and error decay



Conclusions and perspectives

- ▶ Degenerate parabolic problems
- ▶ Euler implicit approach
- ▶ Adaptive linearization scheme:
 - ▶ robust scheme (no restrictions on the initial guess);
 - ▶ mild condition on the time step;
 - ▶ any space discretisation (M)FEM, FV, MPFA, dG, GDM... and mesh.
- ▶ Adaptive space-time discretisation & linearisation
- ▶ Rigorous convergence results, degenerate cases (fast and slow diffusion)
- ▶ More complex models, multi-phase flows, non-equilibrium
- ▶ Domain decomposition
- ▶ Mixed-dimensional models

Joint work with

A. Javed, Hasselt
K. Mitra, Eindhoven
J. Droniou, Montpellier
M. Khan, Melbourne
C. Cancès, Lille

F. Radu, Bergen
K. Kumar, Bergen
J. Both, Bergen
J. Nordbotten, Bergen

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A. Javed, K. Mitra and I. S. Pop,
Math. Comp., <https://doi.org/10.1090/mcom/4235>
<https://arxiv.org/abs/2607.05714>



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