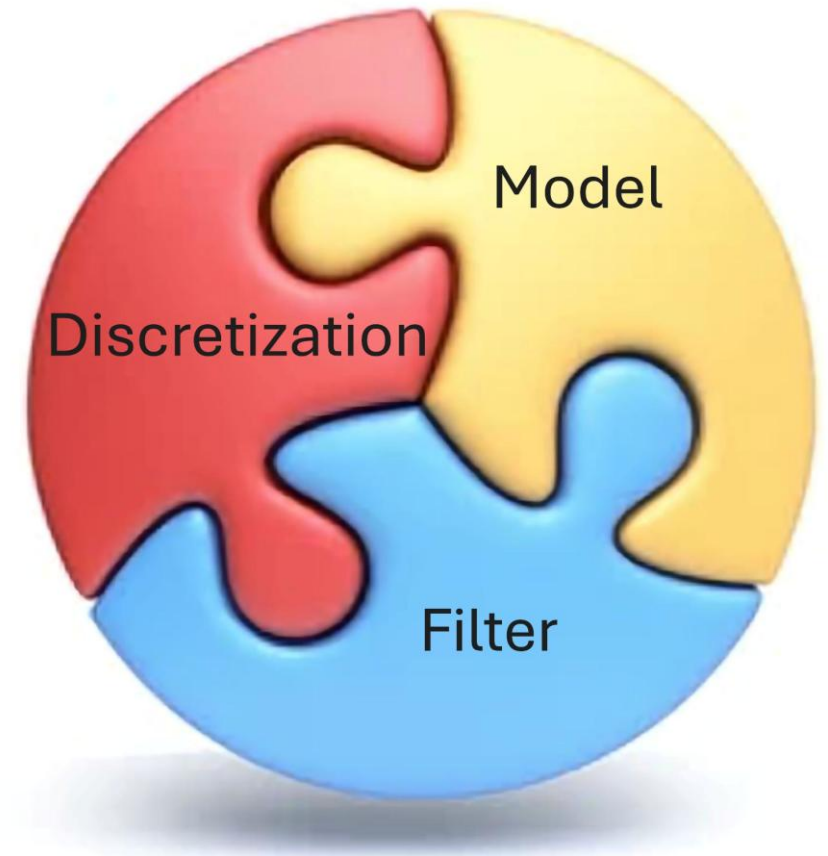


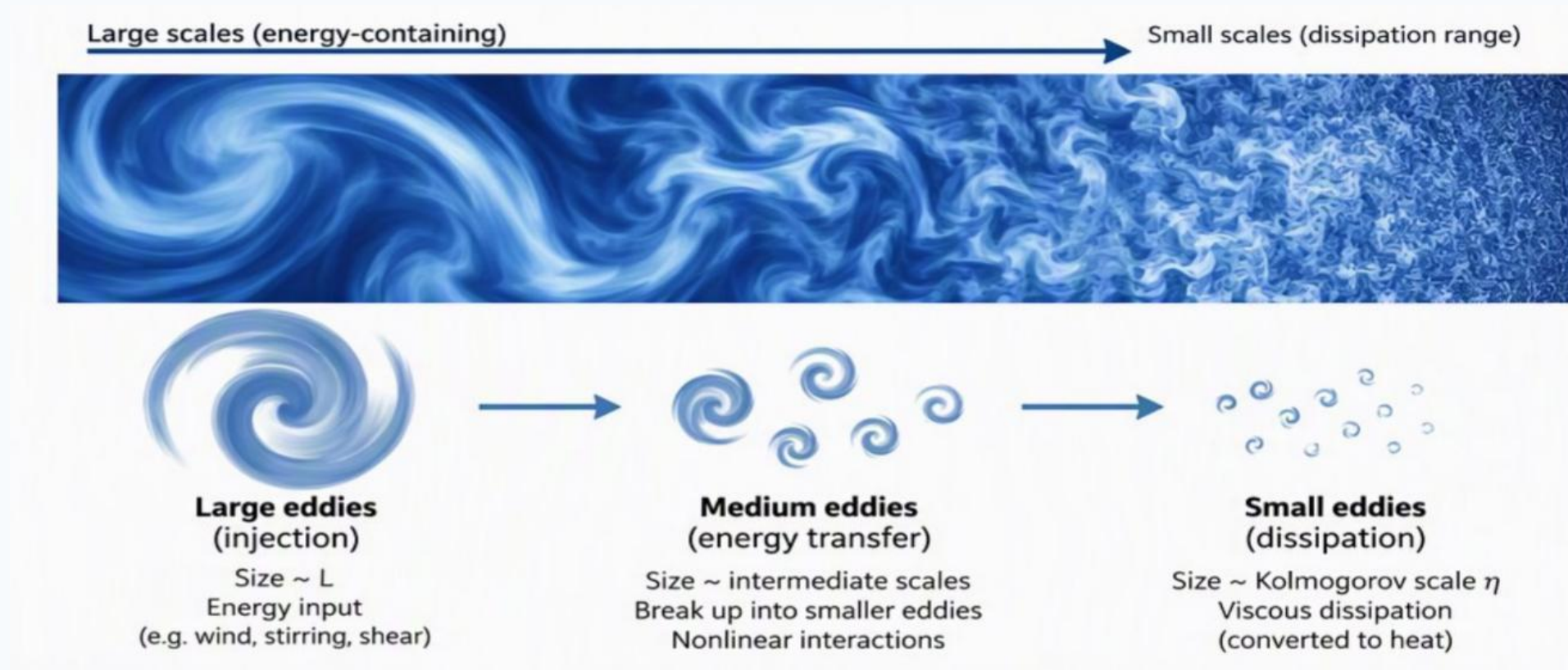
Merging Filtering, Modeling and Discretization

A unified finite-volume framework for
large-eddy simulation of turbulence

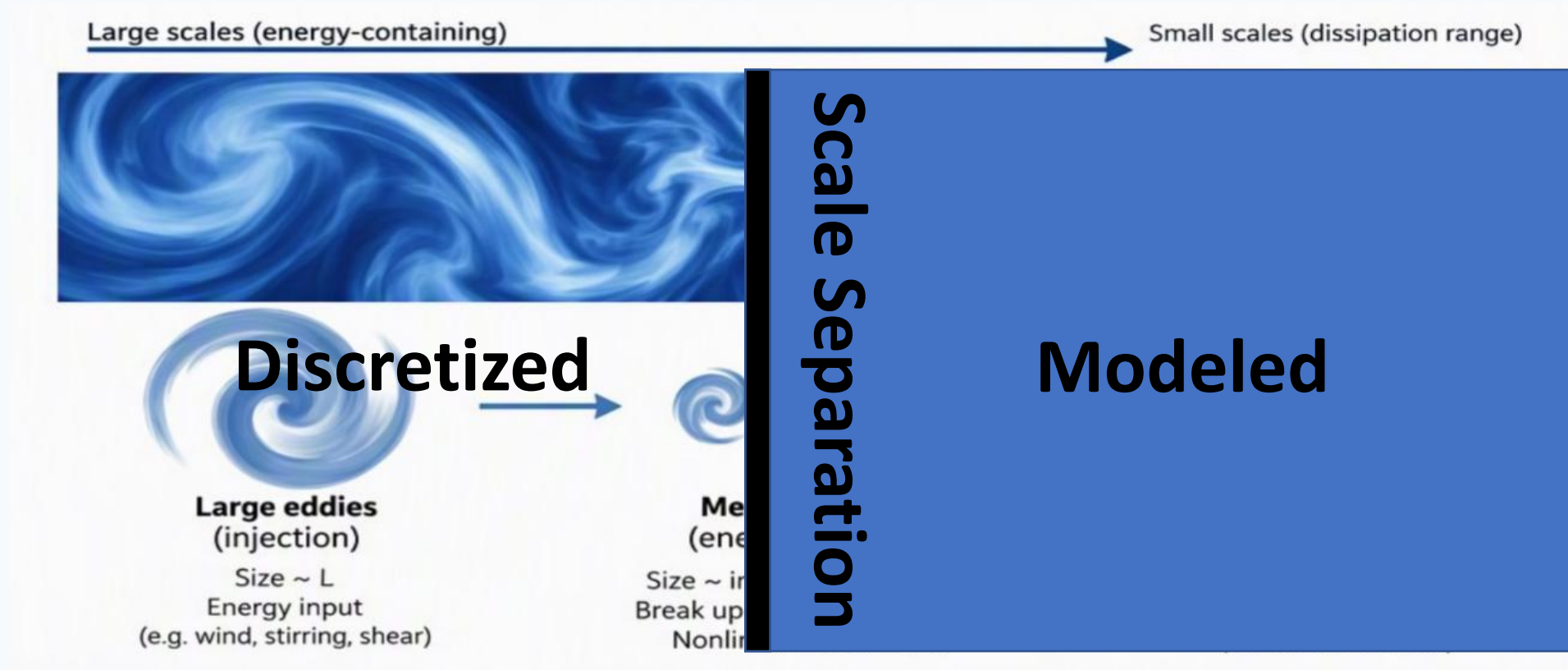
Roel Verstappen

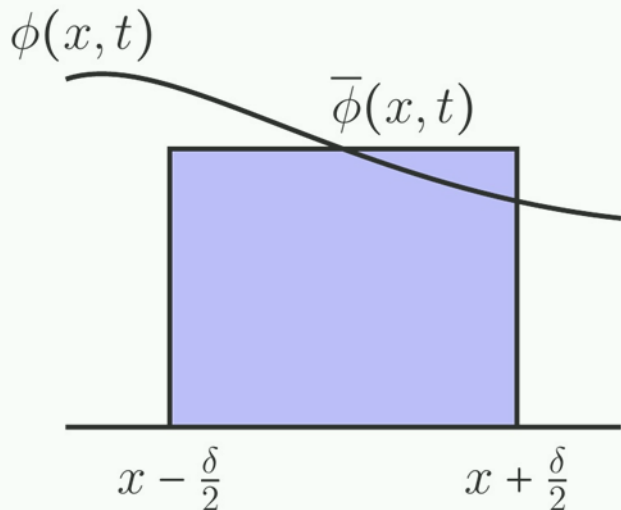


Turbulence: The Energy Cascade



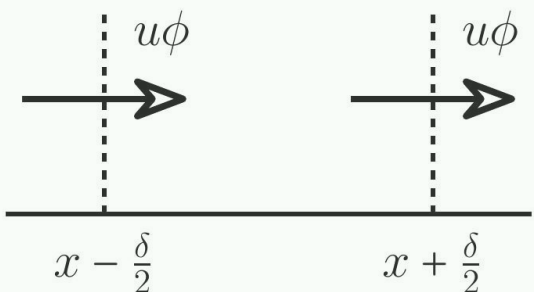
Large-Eddy Simulation



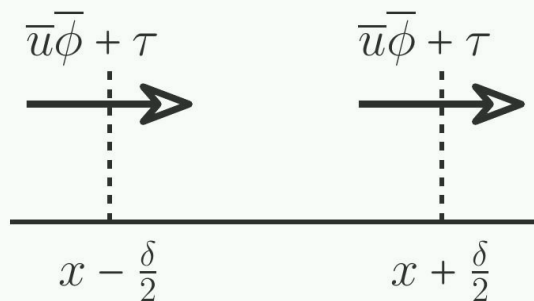


$$\bar{\phi}(x, t) = \frac{1}{\delta} \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} \phi(x, t) dx$$

$$\delta \partial_t \bar{\phi}(x, t) = (u\phi)(x - \frac{\delta}{2}, t) - (u\phi)(x + \frac{\delta}{2}, t)$$

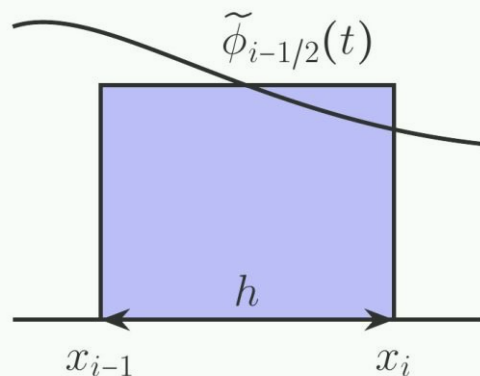


Closure

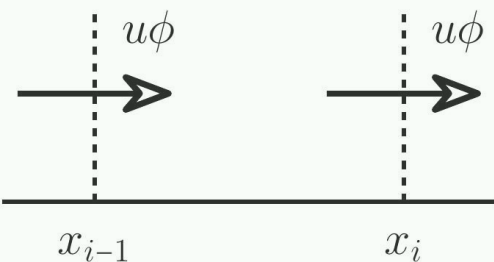


$$\delta \partial_t \bar{\phi}(x, t) = (\bar{u}\bar{\phi} + \tau)(x - \frac{\delta}{2}, t) - (\bar{u}\bar{\phi} + \tau)(x + \frac{\delta}{2}, t)$$

$$\tau(\bar{u}, \bar{\phi}) \simeq u\phi - \bar{u}\bar{\phi}$$

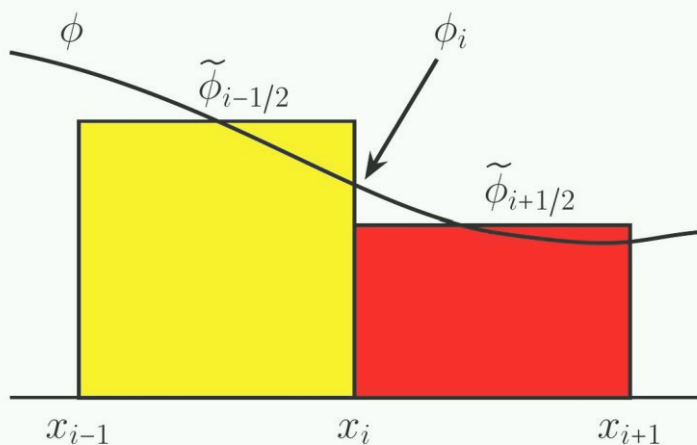


$$\tilde{\phi}_{i-1/2}(t) = \frac{1}{h} \int_{x_{i-1}}^{x_i} \phi(x, t) dx$$



$$h \partial_t \tilde{\phi}_{i-1/2} = (u\phi)_{i-1} - (u\phi)_i$$

Interpolation

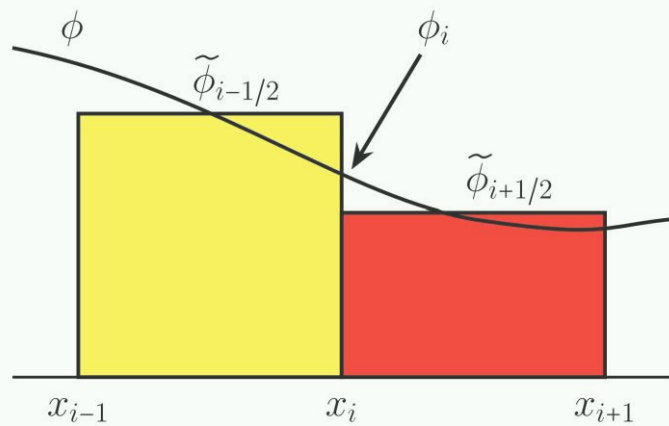


$$\phi_i \simeq \frac{1}{2} (\tilde{\phi}_{i-1/2} + \tilde{\phi}_{i+1/2})$$

$$= \frac{1}{2h} \int_{x_{i-1}}^{x_{i+1}} \phi(x, t) dx = \bar{\phi}_i$$

Two filters, lengths h and $\delta = 2h$

Interpolation



$$\overline{\phi}_i = \frac{1}{2}(\tilde{\phi}_{i-1/2} + \tilde{\phi}_{i+1/2})$$

Filter-based FVM

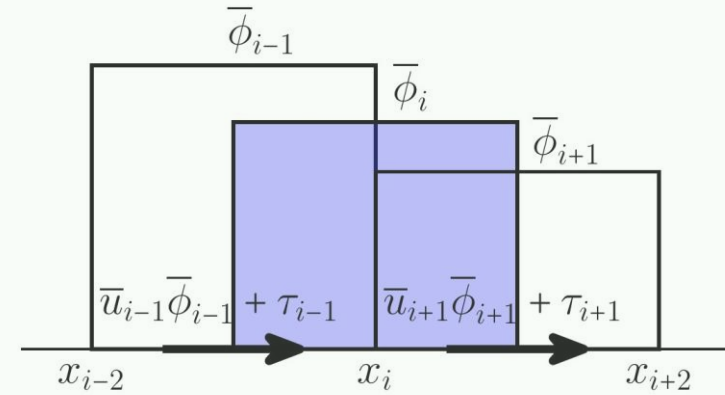
Truncation error

$$h \partial_t \tilde{\phi}_{i-1/2} + \overline{u}_i \overline{\phi}_i - \overline{u}_{i-1} \overline{\phi}_{i-1} = -\tau_i + \tau_{i-1}$$

FVM

Adding neighbours

Truncation error



$$\begin{aligned}
 h \partial_t \tilde{\phi}_{i-1/2} + \bar{u}_i \bar{\phi}_i - \bar{u}_{i-1} \bar{\phi}_{i-1} &= -\tau_i + \tau_{i-1} \\
 h \partial_t \tilde{\phi}_{i+1/2} + \bar{u}_{i+1} \bar{\phi}_{i+1} - \bar{u}_i \bar{\phi}_i &= -\tau_{i+1} + \tau_i +
 \end{aligned}$$

$$\delta \partial_t \bar{\phi}_i + \bar{u}_{i+1} \bar{\phi}_{i+1} - \bar{u}_{i-1} \bar{\phi}_{i-1} = -\tau_{i+1} + \tau_{i-1}$$

FVM

$$\delta \partial_t \bar{\phi}_i + \bar{u}_{i+1} \bar{\phi}_{i+1} - \bar{u}_{i-1} \bar{\phi}_{i-1} =$$

Truncation

$$- \tau_{i+1} + \tau_{i-1}$$

LES

$$\delta \partial_t \bar{\phi}(x) + (\bar{u} \bar{\phi})(x + \frac{\delta}{2}) - (\bar{u} \bar{\phi})(x - \frac{\delta}{2}) =$$

$$- \tau(x + \frac{\delta}{2}) + \tau(x - \frac{\delta}{2})$$

Closure

FVM

$$\delta \partial_t \bar{\phi}_i + \bar{u}_{i+1} \bar{\phi}_{i+1} - \bar{u}_{i-1} \bar{\phi}_{i-1} =$$

$$\text{Truncation}$$

$$- \tau_{i+1} + \tau_{i-1}$$

$$\delta \partial_t \bar{\phi}(x) + (\bar{u} \bar{\phi})(x + \frac{\delta}{2}) - (\bar{u} \bar{\phi})(x - \frac{\delta}{2}) =$$

$$-\tau(x + \frac{\delta}{2}) + \tau(x - \frac{\delta}{2})$$

Closure

No difference between truncation error and closure model if

$$x = x_i \quad \delta = 2h$$

Viscous flux

$$\partial_x \tilde{\phi} = \frac{\phi(x + \frac{h}{2}, t) - \phi(x - \frac{h}{2}, t)}{h}$$

$$\partial_x \tilde{\tilde{\phi}} = \frac{\tilde{\phi}(x + \frac{h}{2}, t) - \tilde{\phi}(x - \frac{h}{2}, t)}{h}$$

$$\nu \partial_x \phi = \nu \partial_x \tilde{\tilde{\phi}} + \nu \partial_x (\phi - \tilde{\tilde{\phi}})$$

$$= \nu \frac{\tilde{\phi}(x + \frac{h}{2}, t) - \tilde{\phi}(x - \frac{h}{2}, t)}{h} + \underbrace{\nu \partial_x (\phi - \tilde{\tilde{\phi}})}$$

unclosed term $\rightarrow \tau$

subgrid modeling and discretization are intrinsically linked

Two filters

$$\tilde{\phi}_{i-1/2}(t) = \frac{1}{h} \int_{x_{i-1}}^{x_i} \phi(x, t) dx \quad \bar{\phi}_i(t) = \frac{1}{\delta} \int_{x_{i-1}}^{x_{i+1}} \phi(x, t) dx$$

Flux

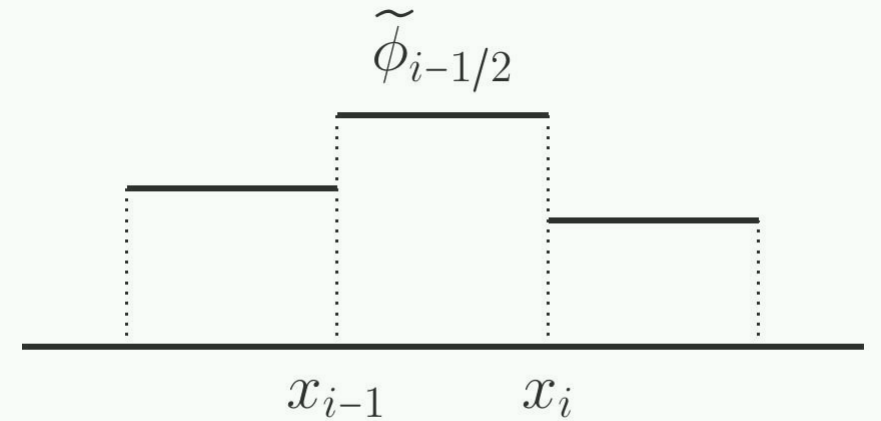
$$\bar{u}_i \bar{\phi}_i - \nu \frac{\tilde{\phi}_{i+1/2} - \tilde{\phi}_{i-1/2}}{h} + \tau_i$$

LES model

error FVM

Grid filter

$$\tilde{\phi}(x, t) = \tilde{\phi}_{i-1/2}(t) \quad \text{for} \quad x_{i-1} \leq x < x_i$$



Idempotent

$$\tilde{\tilde{\phi}} = \tilde{\phi} \quad \implies \quad \tilde{\phi}' = \overline{\phi - \tilde{\phi}} = 0$$

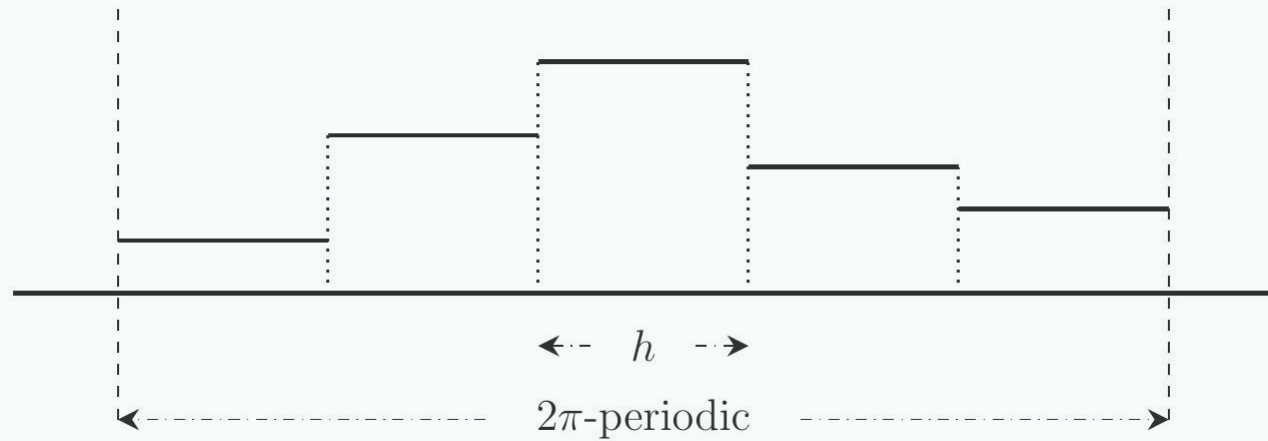
$$\implies \quad (\tilde{\psi}, \phi') = (\psi, \tilde{\phi}') = 0$$



Orthogonal decomposition of function space

Finite volume space

$$\tilde{\mathcal{F}}_h(0, 2\pi)$$



$$\tilde{u} \in \tilde{\mathcal{F}}_h(0, 2\pi) \text{ and } \tilde{\phi} \in \tilde{\mathcal{F}}_h(0, 2\pi) \implies \tilde{u} \tilde{\phi} \in \tilde{\mathcal{F}}_h(0, 2\pi)$$



$\implies$ sub-grid scales are not produced

$$d_t ||\phi||^2 = d_t ||\tilde{\phi}||^2 + d_t ||\phi'||^2 = d_t ||\tilde{\phi}||^2$$

Large and small super-grid scales

$$\tilde{\phi}_{i-1/2} = \underbrace{\frac{1}{2}(\tilde{\phi}_{i+1/2} + \tilde{\phi}_{i-1/2})}_{\bar{\phi}_i} + \underbrace{\frac{1}{2}(\tilde{\phi}_{i-1/2} - \tilde{\phi}_{i+1/2})}_{\phi_i^*}$$

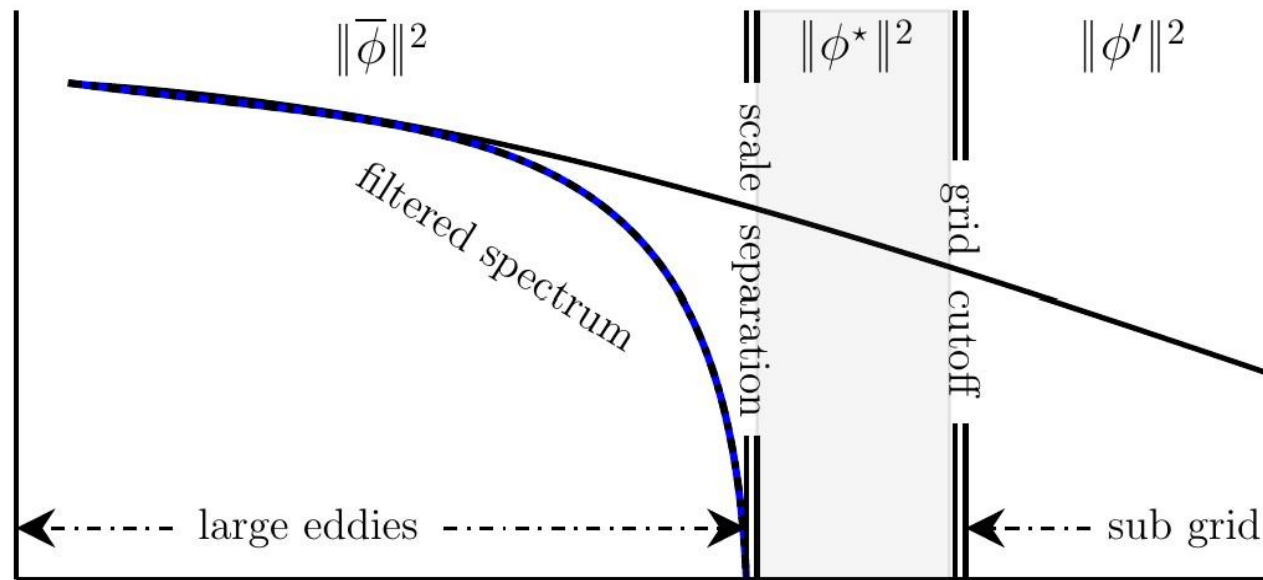
Orthogonal decomposition $\bar{\phi} \cdot \phi^* = 0$

Dissipation $d_t ||\phi||^2 = d_t ||\tilde{\phi}||^2 = d_t ||\bar{\phi}||^2 + d_t ||\phi^*||^2$

Modeling assumption: convection-diffusion balance

$$d_t \|\phi^*\|^2 = 0$$

$$\implies d_t \|\phi\|^2 = d_t \|\bar{\phi}\|^2 + \cancel{d_t \|\phi^*\|^2}$$



Small-scale finite-volume dynamics

$$\partial_t \tilde{\phi} = -\nabla_h \cdot \underbrace{(\Phi + \tau)}_{\text{flux}}$$

$$\phi^* = S \tilde{\phi}$$

$$S = \frac{1}{2} \begin{pmatrix} \diagdown & & \\ & 1 & -1 \\ & & \diagdown \end{pmatrix}$$

$$\partial_t \phi^* = -S \nabla_h \cdot (\Phi + \tau)$$

Modeling small-scale finite-volume dynamics

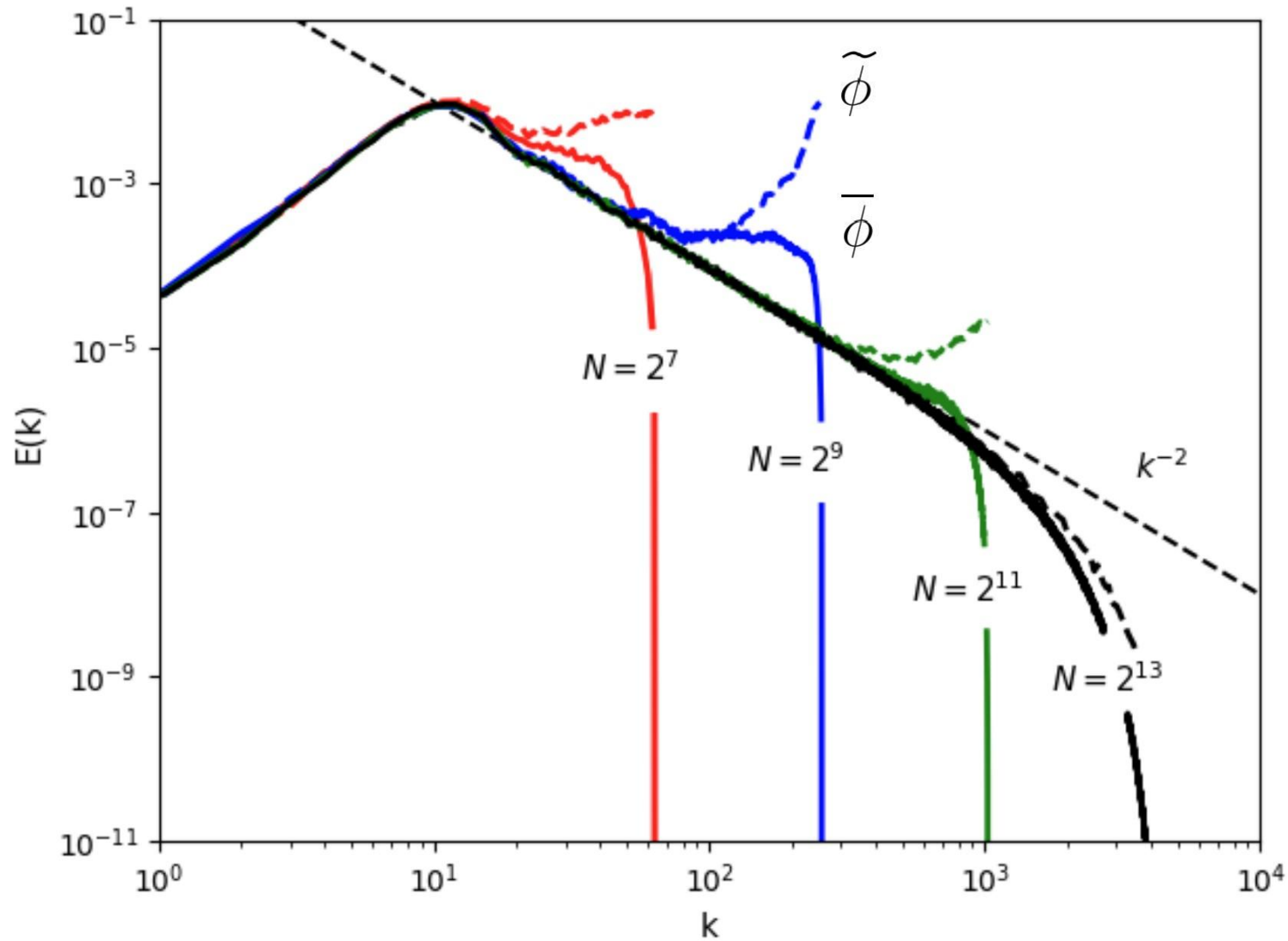
$$\partial_t \phi^* = -S \nabla \cdot (\Phi + \tau)$$

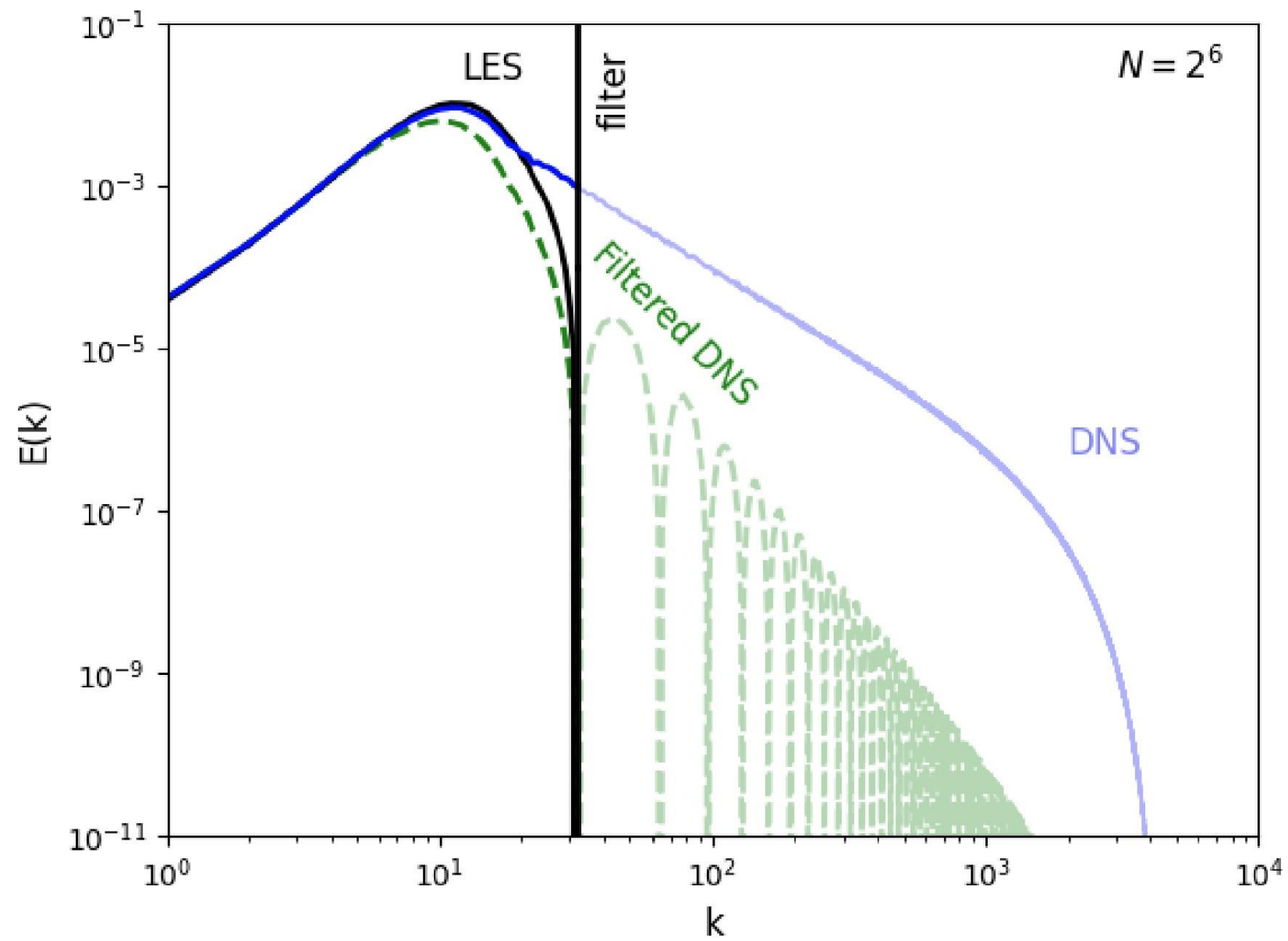
$$\partial_t \phi^* \cdot \phi^* = -S \nabla \cdot (\Phi + \tau) \cdot S \tilde{\phi} = (\Phi + \tau) \cdot \nabla S^\top S \tilde{\phi} = 0$$

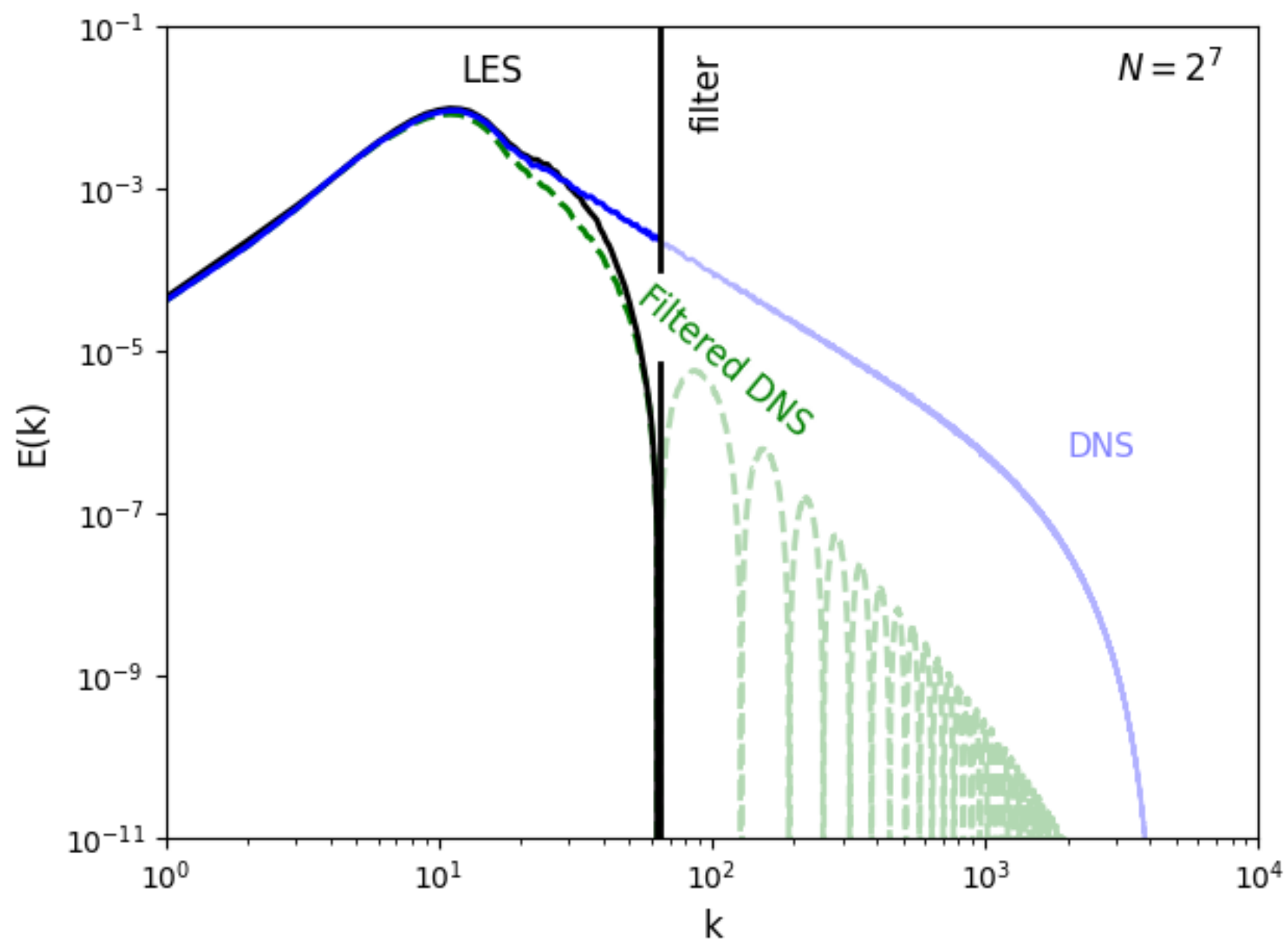
$$\tau = -\alpha \nabla S^\top S \tilde{\phi}$$

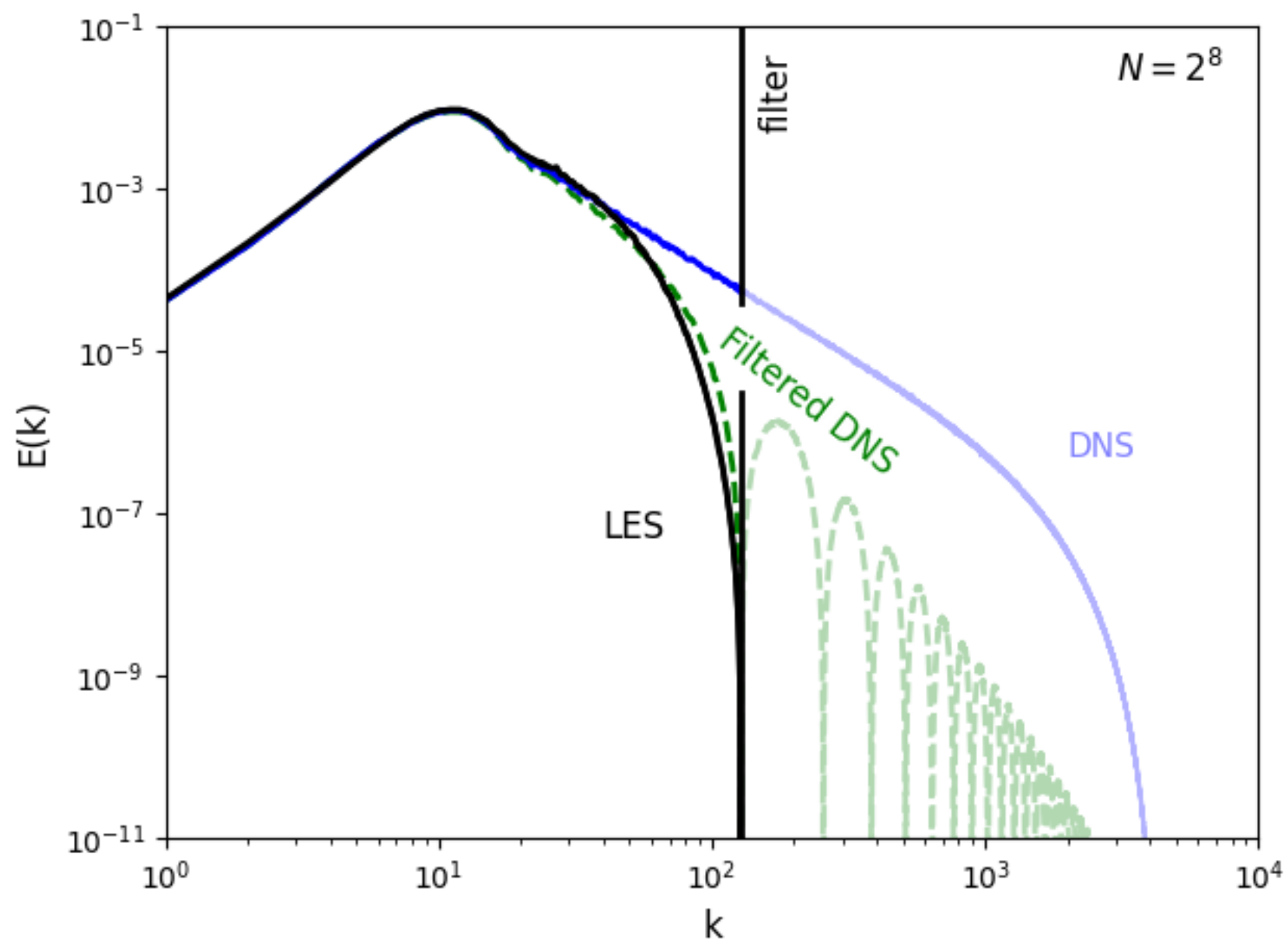
$$\alpha = \frac{\nabla S^\top S \tilde{\phi} \cdot \Phi}{\|\nabla S^\top S \tilde{\phi}\|^2}$$

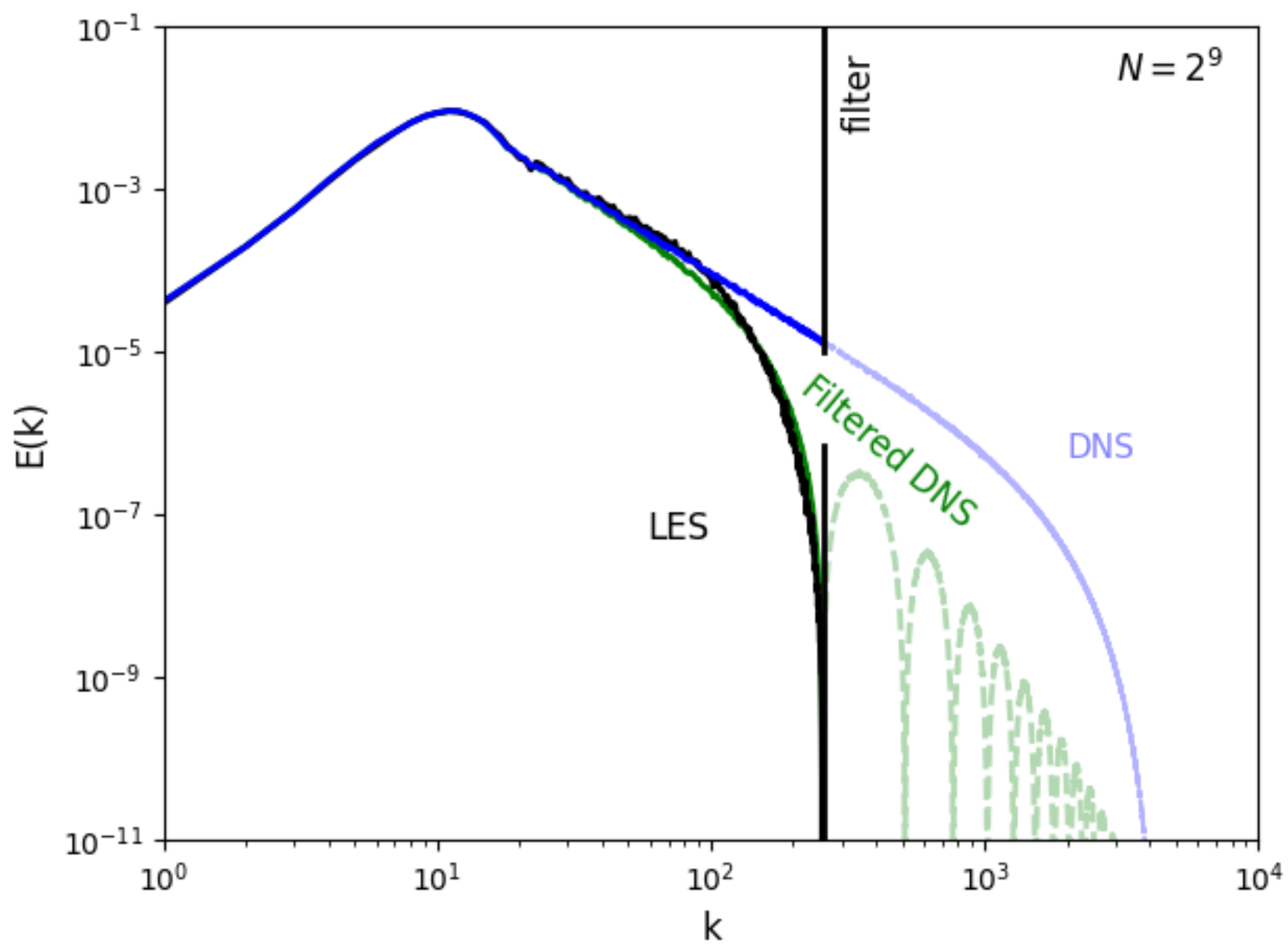
Decaying Burgers' turbulence ($\phi = u$) without model

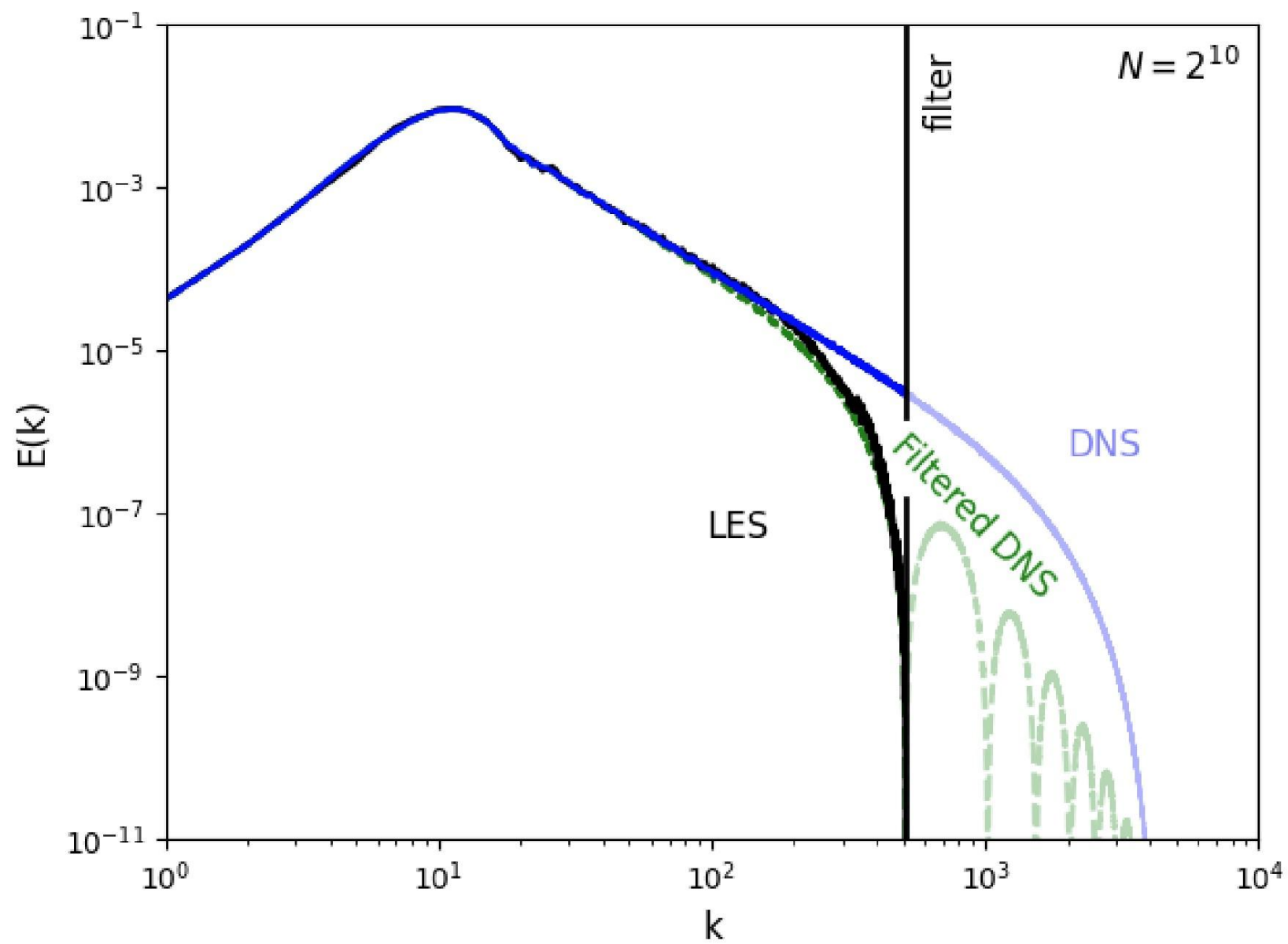












Theoretical Foundation: Bounded Variation (BV)

In the nonlinear regime, L^2 -stability is too weak to imply convergence:

$$L^2 - \text{stability} + \text{consistency} \not\Rightarrow \bar{\phi}_h \longrightarrow \bar{\phi}$$

Energy-stability does not control the redistribution of energy in oscillatory modes

Variation-control is the missing ingredient:

$$\forall n : \underbrace{\sum_{i=1}^N |\tilde{\phi}_{i+1/2}^n - \tilde{\phi}_{i-1/2}^n|}_{\text{Bounded Variation}} \leq C + \text{consistency} + \dots \implies \bar{\phi}_h \longrightarrow \phi$$

Theoretical Foundation: Bounded Variation (BV)

In the nonlinear regime, L^2 -stability is too weak to imply convergence:

$$L^2 - \text{stability} + \text{consistency} \not\Rightarrow \bar{\phi}_h \longrightarrow \bar{\phi}$$

Energy-stability does not control the redistribution of energy in oscillatory modes

Variation-control is the missing ingredient:

$$\forall n : \underbrace{\|\phi_h^{*n}\|_1 \leq \text{Constant}} + \text{consistency} + \dots \implies \bar{\phi}_h \longrightarrow \bar{\phi}$$

BV: bound on ϕ_h^* controls oscillations

Unified Mathematical-Physical Interpretation

Physics of Turbulence:

Energy and its dissipation: L_2 -norm

Maths of Nonlinear PDEs:



Bounded Variation: L_1 -norm

Bounded-Variation Model

$$d_t \|\phi^\star\|_2^2 = 0$$

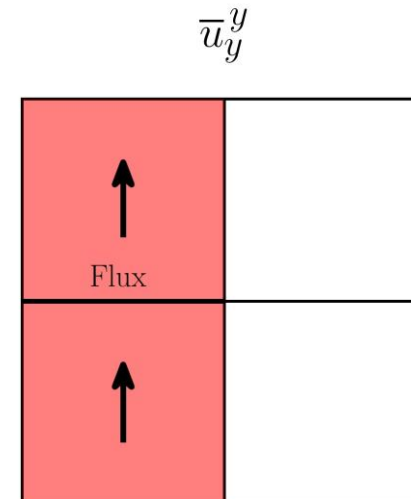
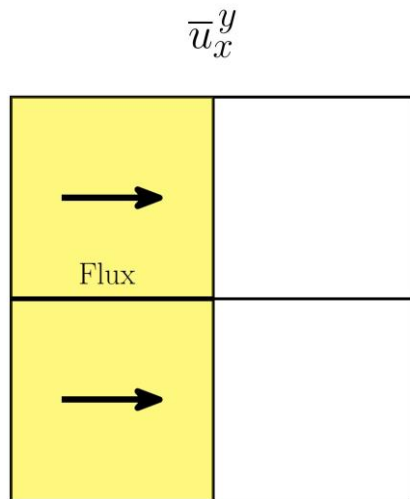
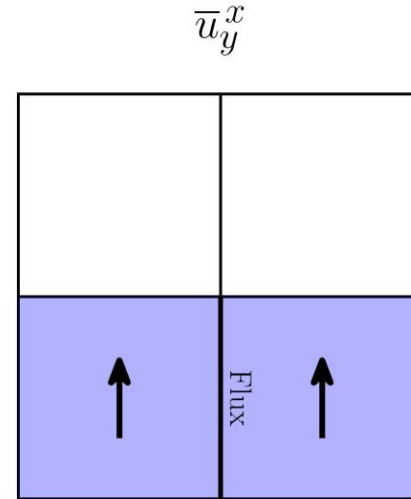
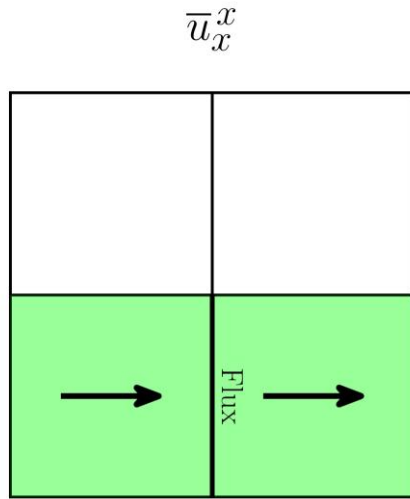


$$\begin{aligned} \|\phi^{\star n}\|_1 &\leq \sqrt{N} \|\phi^{\star n}\|_2 = \sqrt{N} \|\phi^{\star 0}\|_2 \\ &= \sqrt{N} \left(\sum_{i=1}^N \frac{1}{4} (\tilde{\phi}_{i+1/2}^0 - \tilde{\phi}_{i-1/2}^0)^2 \right)^{1/2} \\ &\leq \sqrt{N} \sqrt{\frac{1}{4} N h^2 (\sup \partial_x \tilde{\phi}^0)^2} \end{aligned}$$



$$\|\phi^{\star n}\|_1 \leq \text{Const}$$

3D generalization using anisotropic interpolation filters



Tensorial interpolation filter

$$\begin{pmatrix} \overline{u}_x^x & \overline{u}_y^x \\ \overline{u}_x^y & \overline{u}_y^y \end{pmatrix}$$

Tensorial filters

$$\begin{pmatrix} \tilde{u}_x & \tilde{u}_y \\ \tilde{u}_x & \tilde{u}_y \end{pmatrix} = \begin{pmatrix} \overline{u}_x^x & \overline{u}_y^x \\ \overline{u}_x^y & \overline{u}_y^y \end{pmatrix} + \begin{pmatrix} u_x^{\star x} & u_y^{\star x} \\ u_x^{\star y} & u_y^{\star y} \end{pmatrix}$$

Tensorial filter

$$\begin{pmatrix} \widetilde{u}_x & \widetilde{u}_y \\ \widetilde{u}_x & \widetilde{u}_y \end{pmatrix} = \begin{pmatrix} \overline{u}_x^x & \overline{u}_y^x \\ \overline{u}_x^y & \overline{u}_y^y \end{pmatrix} + \begin{pmatrix} u_x^{\star x} & u_y^{\star x} \\ u_x^{\star y} & u_y^{\star y} \end{pmatrix}$$

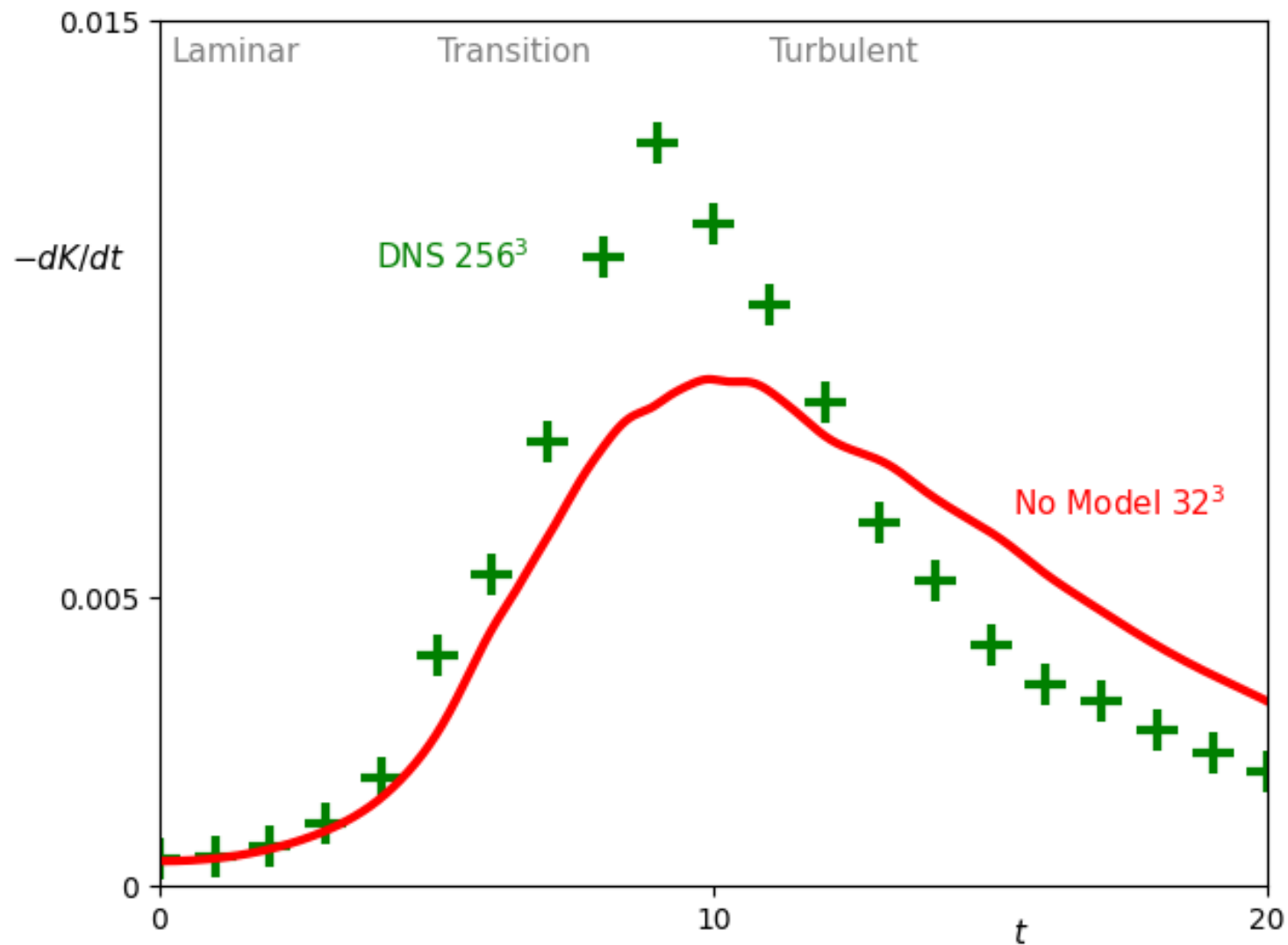
$$\widetilde{u} = \overline{u} + u^{\star}$$

Grid filter

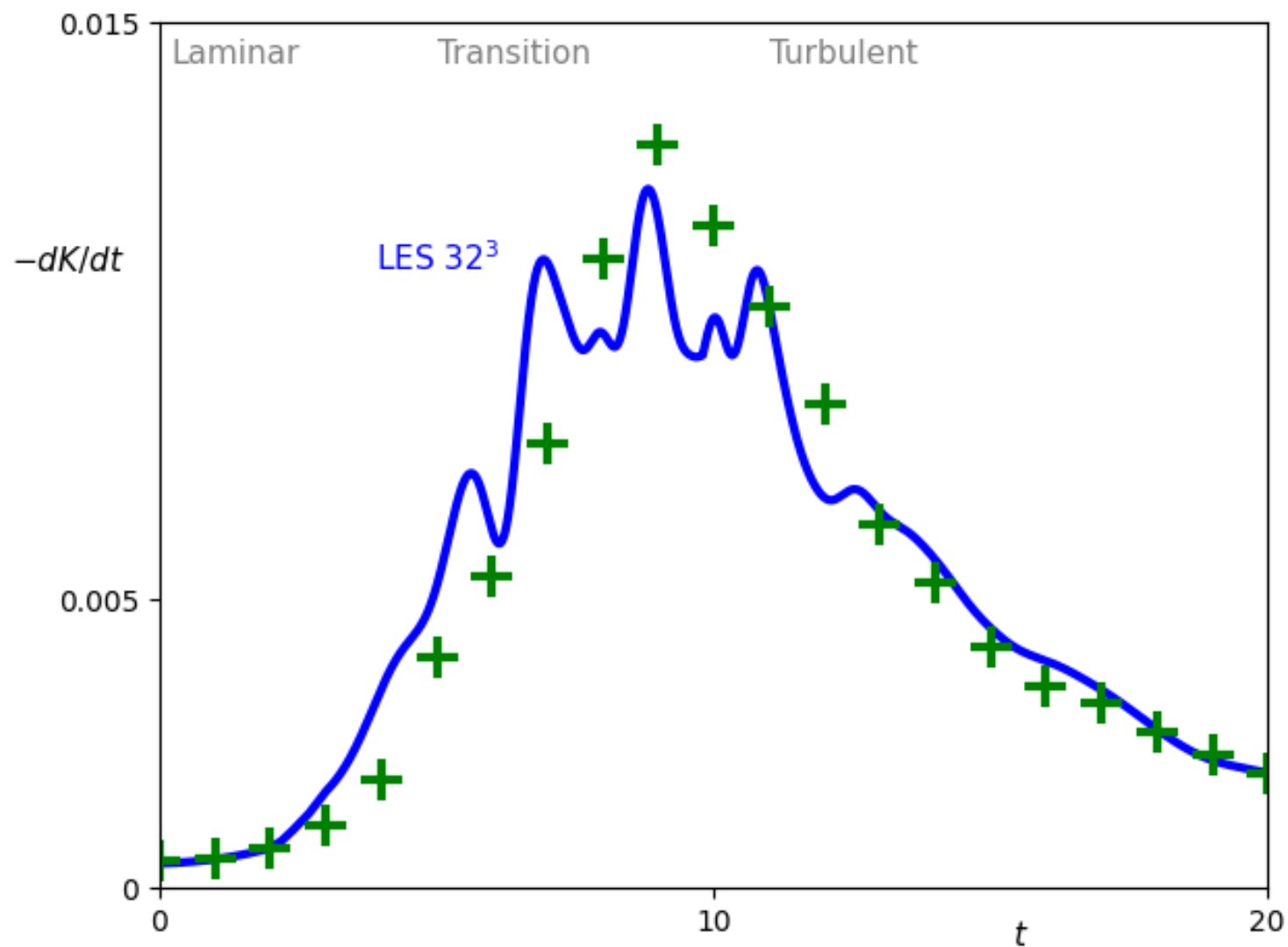
Large eddies

Small scales

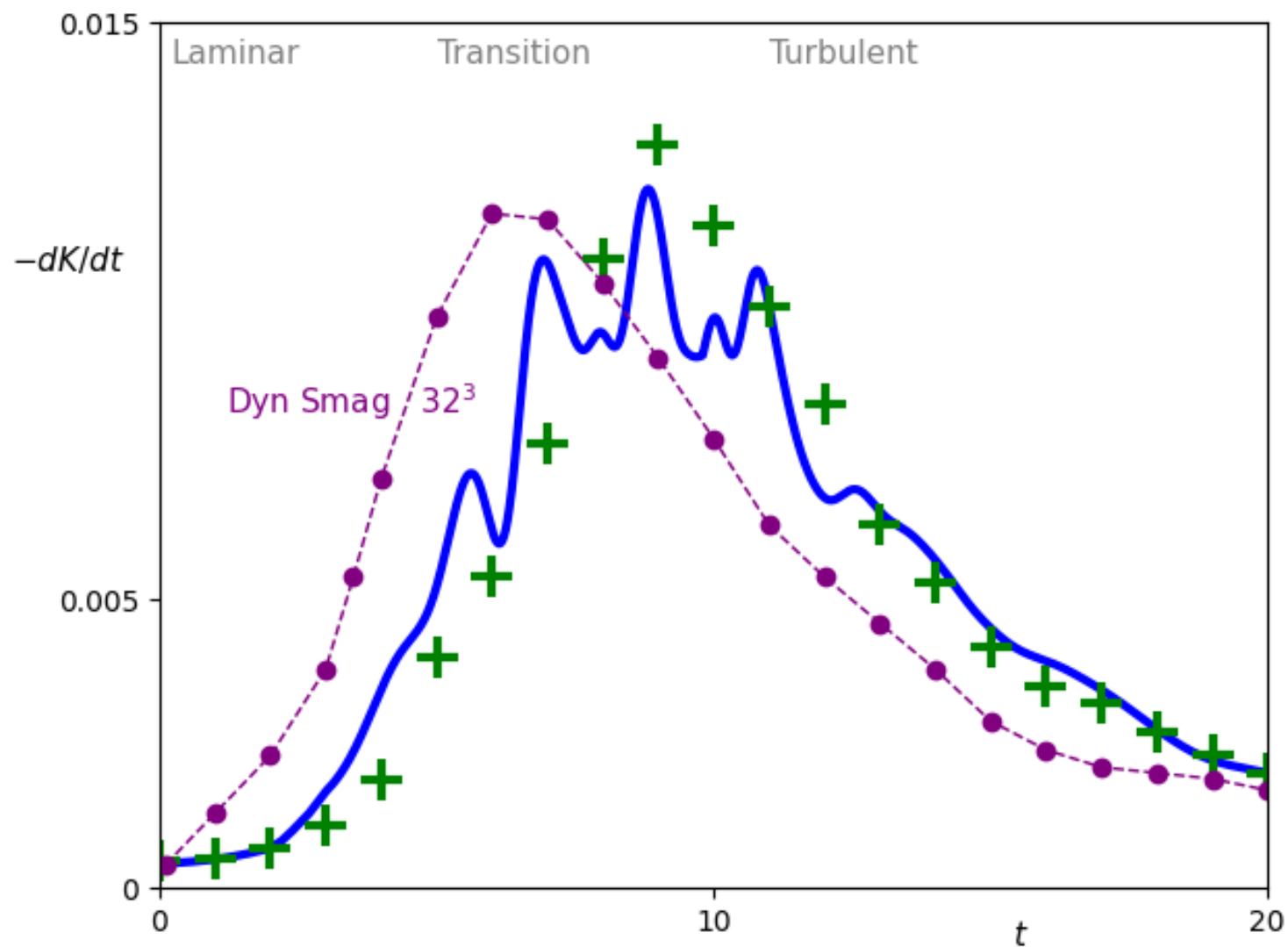
Dissipation 3D Taylor-Green Vortex Re=1600



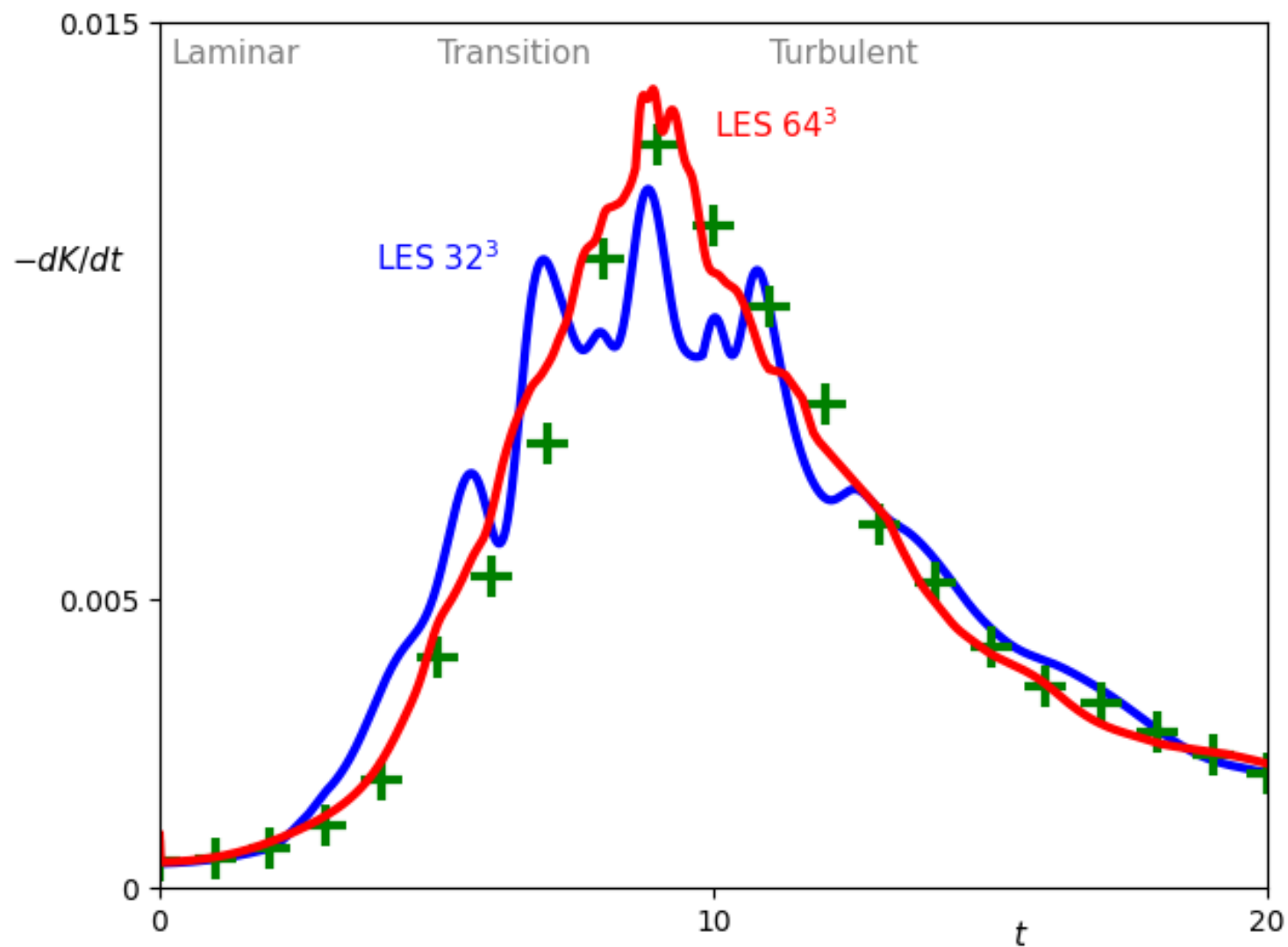
Dissipation Taylor-Green Vortex Re=1600



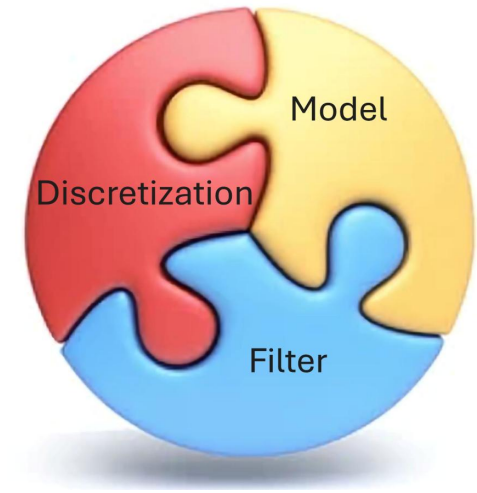
Dissipation Taylor-Green Vortex Re=1600



Dissipation Taylor-Green Vortex Re=1600



- 1 Filtering, modeling and discretization are integrated
- 2 Closure and truncation become equivalent in FVM-based LES
- 3 BV modeling provides nonlinear stability and convergence
- 4 Pathway towards unified 3D LES



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<https://doi.org/10.1007/s10494-026-00789-1>

RESEARCH

Merging Filtering, Modeling and Discretization to Simulate Large Eddies In Burgers' Turbulence

Roel Verstappen¹

