

Efficient numerical solution of PIDEs for electricity derivatives under jump dynamics

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Swing options

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Infinite-activity Lévy models

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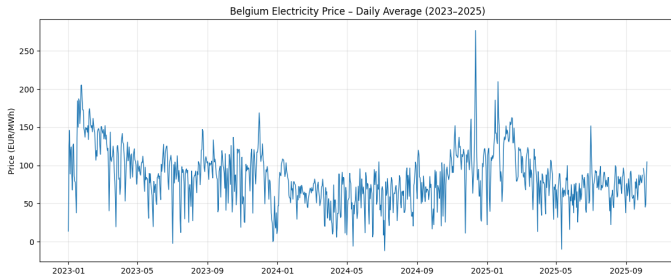
Swing options in electricity

- ▶ two-factor mean-reverting model with jumps,
- ▶ repeated exercise with local/global volume constraints,
- ▶ convection-dominated PIDEs with a nonlocal integral.

European options on two assets

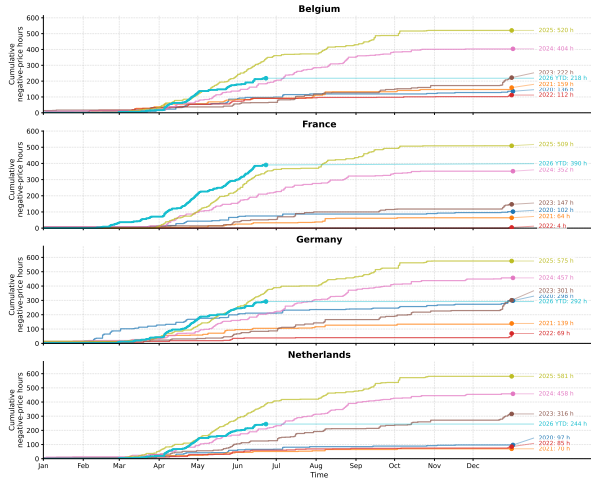
- ▶ exponential Lévy dynamics,
- ▶ infinitely many small jumps,
- ▶ a singular two-dimensional nonlocal integral.

Which model for the underlying spot price?



Electricity spot prices typically exhibit

- ▶ **mean reversion,**
- ▶ sudden upward and downward **spikes,**
- ▶ daily, weekly and yearly **seasonality.**



Negative prices occur in several European day-ahead electricity markets.

	spot price model	exercise right	valuation method
Jaillet <i>et al.</i> (2004)	mean-reverting diffusion	Bermudan, fixed dates	lattice
Zhang & Oosterlee (2013)	Lévy, char. function known	Bermudan, fixed dates	COS method
Wilhelm & Winter (2008)	mean-reverting diffusion	American + fixed waiting time	Finite element
Dahlgren (2005)	mean-reverting diffusion	American + waiting time	Finite difference
Benth <i>et al.</i> (2011), Eriksen <i>et al.</i> (2014)	mean-reverting Lévy	continuous exercise	Finite difference
Kjaer (2008)	mean-reverting diffusion + jumps	Bermudan, fixed dates	Finite difference
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Under a pricing measure $\mathbb{Q}$,

$$S_t = X_t + Y_t,$$

with

$$dX_t = \alpha(\mu - X_t) dt + \sigma dW_t, \quad dY_t = -\beta Y_t dt + J_t dN_{\lambda,t}.$$

Regular factor X

- ▶ mean-reversion level μ ,
- ▶ mean-reversion speed α ,
- ▶ volatility σ .
- ▶ Seasonality incorporated by $\mu(t)$.
- ▶ The model permits negative prices.

Spike factor Y

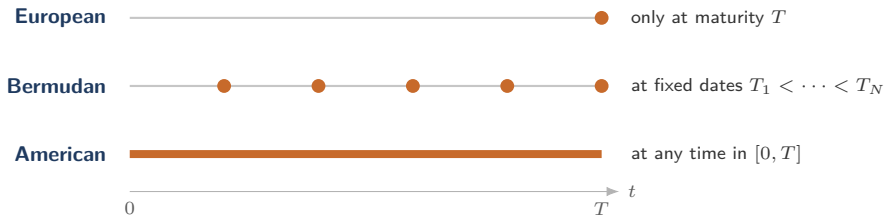
- ▶ mean-reversion speed β ,
- ▶ jump intensity λ ,
- ▶ jump-size density f .

Which type of option?

An **option** gives its holder the right — not the obligation — to buy (*call*) or sell (*put*) an underlying asset at a fixed *strike* K . Exercising at time t pays

$$\text{call: } (S_t - K)^+, \quad \text{put: } (K - S_t)^+.$$

When may the holder exercise?



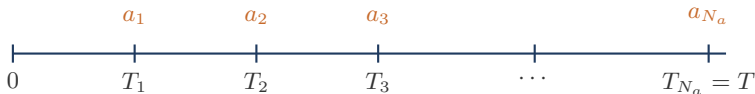
Swing option: multiple exercise dates

	spot price model	exercise right	valuation method
Jaillet <i>et al.</i> (2004)	mean-reverting diffusion	Bermudan, fixed dates	lattice
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At predetermined action times

$$T_n = n\Delta T, \quad n = 1, \dots, N_a,$$

the holder chooses the amount a_n to purchase.



Local constraint

$$0 \leq a_n \leq L.$$

At a single action time, at most L units can be purchased.

Global constraint

$$\sum_{n=1}^{N_a} a_n \leq M.$$

The cumulative purchased volume cannot exceed M .

Let Z_t denote the amount of energy purchased before time t .

The exercise decisions satisfy

$$a_n \in \{0, 1, \dots, L\}, \quad \sum_{n=1}^{N_a} a_n \leq M.$$

At an action time T_i , the swing-option value is

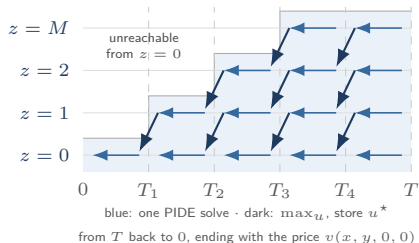
$$v(x, y, z, T_i) = \sup_{u \in \mathcal{A}^{N_a}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{n=i}^{N_a} e^{-r(T_n - T_i)} (X_{T_n} + Y_{T_n} - K) u_{T_n} \mid X_{T_i} = x, Y_{T_i} = y, Z_{T_i} = z \right].$$

Thus, in addition to the two price factors (x, y) , the already-used volume z becomes a state variable.

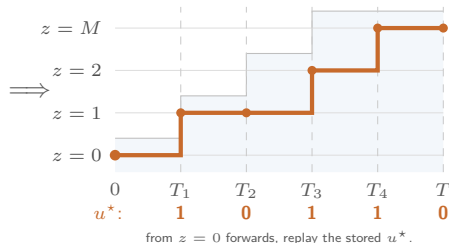
$$\begin{cases} \partial_t v + \frac{\sigma^2}{2} \partial_{xx} v + \alpha(\mu - x) \partial_x v - \beta y \partial_y v - (r + \lambda)v + \lambda \int_{\mathbb{R}} v(x, y + \xi, z, t) f(\xi) d\xi = 0, \\ T_{i-1} < t < T_i \\ \\ v(x, y, z, T_i) = \sup_{u \in \mathcal{A}^{N_a}} \sum_{n=i}^{N_a} e^{-r(T_n - T_i)} \mathbb{E}[(X_{T_n} + Y_{T_n} - K)u_{T_n} \mid X_t = x, Y_t = y, Z_t = z], \end{cases}$$

for $1 \leq i \leq N_a$ and $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times [0, M]$.

1. Backward induction



2. Forward pass — one path



- We consider first the subproblem

$$\partial_t v = \frac{\sigma^2}{2} v_{xx} + \alpha(\mu - x) \partial_x v - \beta y \partial_y v - (r + \lambda)v + \lambda \int_{\mathbb{R}} v(x, y + \xi, t) f(\xi) d\xi,$$
$$v(x, y, 0) = \max(x + y - K, 0).$$

- Then, combined by the dynamic programming, swing option can be priced.

The PIDE has several difficult numerical features:

- ▶ **Convection-dominated behaviour in the x -direction** through

$$\alpha(\mu - x)\partial_x v.$$

- ▶ **Pure convection in the y -direction** through

$$-\beta y \partial_y v.$$

- ▶ A **nonlocal integral term** coupling distant values in the y -direction.

$$\lambda \int_{\mathbb{R}} v(x, y + \xi, z, t) f(\xi) d\xi.$$

- ▶ A nonsmooth initial condition

$$v(x, y, 0) = \max(x + y - K, 0).$$

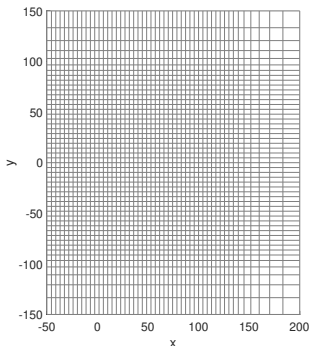
Which numerical method?

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The spatial domain is truncated to

$$[x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}],$$

and a smooth nonuniform Cartesian grid is introduced.



Let

$$V_{i,j}(t) \approx v(x_i, y_j, t).$$

The local differential terms are discretised using

- ▶ second-order finite differences for $\partial_{xx}v$;
- ▶ second-order upwind discretisation, in particular QUICK, for $\partial_x v$ and $\partial_y v$.

The resulting local matrix is

$$A = A^D + A_x + A_y,$$

where

A is sparse.

At a grid point (x_i, y_j) ,

$$\lambda \int_{\mathbb{R}} v(x_i, y_j + \xi, t) f(\xi) d\xi \approx \lambda \sum_{\ell=0}^{m_2} \tilde{B}_{j\ell} V_{i,\ell}(t).$$

The coefficients $\tilde{B}_{j\ell}$ are obtained by integrating a piecewise-linear interpolant of v against the jump density.

Define

$$B = I_1 \otimes \lambda \tilde{B}, \quad \tilde{B} = (\tilde{B}_{j\ell})_{0 \leq j, \ell \leq m_2}.$$

Then

$$\boxed{\frac{dV}{dt} = (A + B)V.}$$

$$\underbrace{A}_{\text{sparse local operator}} + \underbrace{B}_{\text{dense jump operator}}.$$

The dense operator B is treated by fixed-point iteration, so that no linear system involving B has to be solved. For $t_n = n\Delta t$ and $V^n \approx V(t_n)$.

Crank–Nicolson with fixed-point iteration

$$\left(I - \frac{1}{2}\Delta t A\right) Y_\ell = \left(I + \frac{1}{2}\Delta t A\right) V^n + \frac{1}{2}\Delta t \left(BV^n + BY_{\ell-1}\right),$$

$$V^{n+1} = Y_{\ell_{\max}}.$$

- ▶ implicit solves involve only the sparse matrix A ;
- ▶ Rannacher time stepping is applied at the initial steps.

DIRK with fixed-point iteration

$$W_1 = V^n + (1 - \theta)\Delta t(A + B)V^n, \\ (I - \theta\Delta t A)Y_\ell = W_1 + \theta\Delta t B Y_{\ell-1},$$

$$\widehat{Y} = Y_{\ell_{\max}},$$

$$W_2 = V^n + \frac{1}{2}\Delta t(A + B)V^n + \left(\frac{1}{2} - \theta\right)\Delta t(A + B)\widehat{Y},$$

$$(I - \theta\Delta t A)Z_\ell = W_2 + \theta\Delta t B Z_{\ell-1},$$

$$V^{n+1} = Z_{\ell_{\max}}.$$

- ▶ fixed-point iteration for B at both stages;
- ▶ $\theta = 1 - \frac{1}{\sqrt{2}}$: the scheme is L-stable.

- For CNFI scheme

Theorem

- *CNFI converges to the Crank–Nicolson scheme as $\ell \rightarrow \infty$ when $\Delta t \leq C(\alpha, \beta, \lambda)$.*
- *Crank–Nicolson is unconditionally stable in the ℓ_2 -norm for $\Delta t \leq C(\alpha, \beta, \lambda)$. Moreover, it is second-order convergent under the assumption of a smooth initial condition.*

- For DIRKFI scheme

Theorem

- *DIRKFI converges to the DIRK scheme as $\ell \rightarrow \infty$ when $\Delta t \leq C(\alpha, \beta, \lambda, \theta)$.*
- *DIRK is unconditionally stable in the ℓ_2 -norm for $\theta \in [\frac{1}{4}, \frac{1}{2}]$ and $\Delta t \leq C(\alpha, \beta, \lambda, \theta)$. Moreover, it is second-order convergent under the assumption of a smooth initial condition.*

The jump sizes follow a double-exponential density,

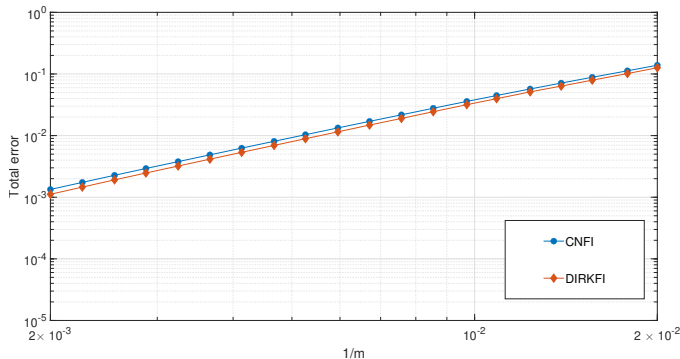
$$f(y) = p\eta_1 e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + (1-p)\eta_2 e^{\eta_2 y} \mathbf{1}_{\{y < 0\}}.$$

μ	α	β	σ	r	λ	p	η_1	η_2	K
80	8	126	11	0.03	52	0.6	0.01	0.02	50

The computational domain is

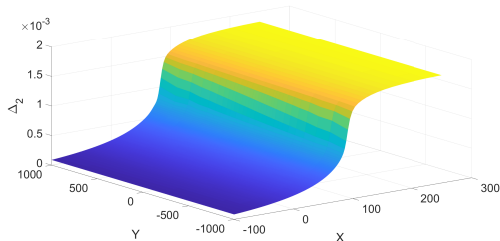
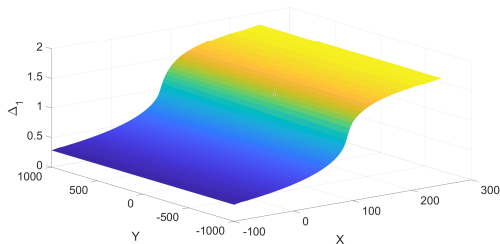
$$x \in [-2K, 5K], \quad y \in [-20K, 20K].$$

The large value of β models rapidly decaying price spikes, while $\lambda = 52$ corresponds approximately to one jump per week.



The total error is measured on a fixed region of interest while space and time are refined simultaneously.

- ▶ CNFI and DIRKFI give very similar errors.
- ▶ The errors decrease regularly under refinement.
- ▶ The observed convergence rate is close to **second order**.



- The two Delta are

$$\Delta_1 = \frac{\partial v}{\partial x}, \quad \Delta_2 = \frac{\partial v}{\partial y}.$$

- Obtained directly from the spatial finite-difference approximations.

- ▶ The swing option problem reduces to a sequence of two-dimensional PIDEs coupled through dynamic programming.

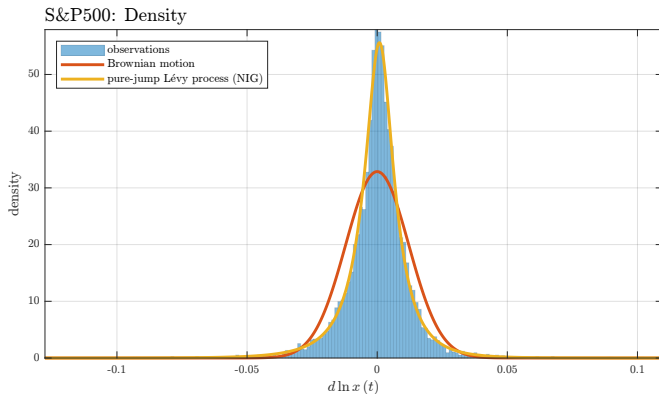
- ▶ Spatial discretisation gives

$$\frac{d}{dt}V = (A + B)V,$$

with a **sparse local operator** A and a **dense jump operator** B .

- ▶ CNFI and DIRKFI keep the dense jump matrix outside the implicit linear systems and show second-order convergence, in theory and experiments.
- ▶ The same computation provides prices, Greeks and optimal exercise policies over the state space.

What changes when the jump process has infinite activity?



Empirical log-return distributions typically show

- ▶ sharper peaks,
- ▶ heavier tails,
- ▶ skewness,

than a Gaussian density.

Motivation

Pure-jump Lévy models can reproduce these features while remaining analytically and numerically tractable.

	one asset (1-D)	two assets (2-D)
finite activity	Almendral & Oosterlee (2005): European options d'Halluin <i>et al.</i> (2005); Toivanen (2008): American options	Clift & Forsyth (2008): European/American Yousuf & Khaliq (2023): American, regime switching + jumps Boen & in 't Hout (2020): American Merton in 't Hout (2025): American Kou + Greeks
infinite activity	Cont & Voltchkova (2005): small jumps $\rightarrow$ diffusion Wang <i>et al.</i> (2007) : CGMY, tailored quadrature + FFT	very little — Fourier methods exist (Jackson <i>et al.</i> , 2008; Ruijter & Oosterlee, 2012) but need the characteristic exponent $\Rightarrow$ this work

We model the price of some stocks as

$$d \ln x_i(t) = \mu_i dt + dL_i(t) \quad (i = 1, 2)$$

where $\mu \in \mathbb{R}^2$ is the drift vector and L is an **infinite activity pure-jump** Lévy process, which is completely characterized by the function $\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$, called **Lévy measure**.

Finite activity

$$\int_{\mathbb{R}^2} \ell(z) dz < \infty.$$

Only finitely many jumps occur on finite time intervals.

Infinite activity

$$\int_{\mathbb{R}^2} \ell(z) dz = \infty.$$

Infinitely many small jumps occur in every finite time interval.

We focus on the **Normal Tempered Stable** (NTS) process. The d -dimensional case is characterized by

$$\ell(z) = AK_{\alpha+1}(B\|z\|_{\rho})\|z\|_{\rho}^{-\alpha-1}e^{\langle\eta,z\rangle_{\rho}},$$

for some positive constant $A, B > 0$, $0 \leq \alpha < 1$, $\eta \in \mathbb{R}^2$, $\rho \in \mathbb{R}^{2 \times 2}$ (symmetric and positive definite), $\|z\|_{\rho} = \sqrt{z^{\top} \rho^{-1} z}$ and $K_{\alpha+1}$ is a modified Bessel function of the second kind. See Barndorff-Nielsen *et al.* (2001).

Variance Gamma

$$\alpha = 0.$$

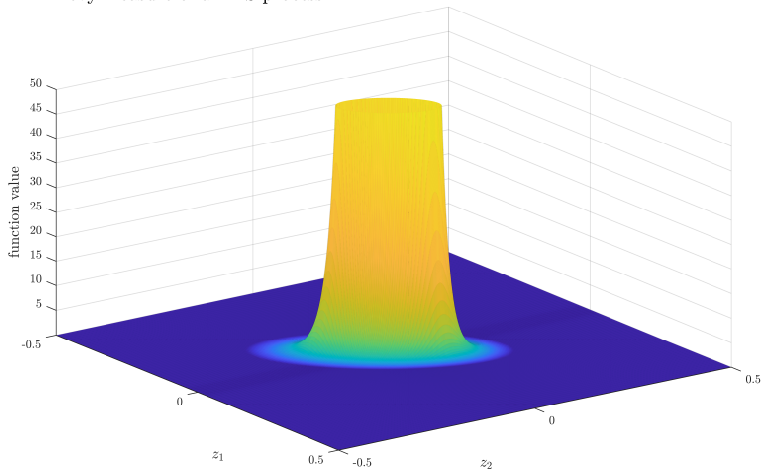
The corresponding process has finite variation.

Normal Inverse Gaussian

$$\alpha = \frac{1}{2}.$$

A more singular infinite-activity case.

Lévy measure of a NTS process



For a European payoff ϕ and maturity T ,

$$P(t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}\phi(X(T)) \mid \mathcal{F}_t] = u(X(t), T - t).$$

The pricing function solves

$$\partial_t u = \mathcal{A}u - ru, \quad u(x, 0) = \phi(x),$$

with the nonlocal part

$$\mathcal{A}u(x, t) = \int_{\mathbb{R}^2} \left[u(x + \gamma(x, z), t) - u(x, t) - \sum_{i=1}^2 \gamma_i(x, z) u_{x_i}(x, t) \right] \ell(z) \, dz,$$
$$\gamma_i(x, z) = x_i(e^{z_i} - 1).$$

New difficulty

The integral is now two-dimensional and its weight $\ell(z)$ is singular near $z = 0$.

The Lévy measure is singular

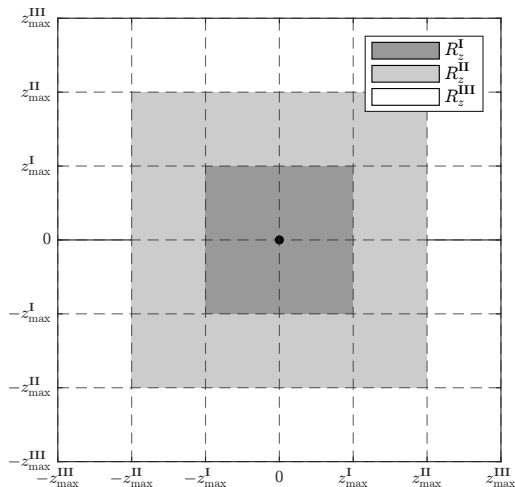
Near $z = 0$, the density is unbounded. A non-standard quadrature is required to recover a useful convergence rate.

The nonlocal operator is dense

Iterative approach for solving the linear system at each time step (fixed-point iteration), combined with an FFT-based algorithm to efficiently accelerate matrix-vector products.

Main contribution

We extend the approach by Wang *et al.* (2007) to the 2-dimensional setting.



Write the truncated jump domain as

$$R_z = R_z^I \cup R_z^{II} \cup R_z^{III}.$$

- R_z^I : **at the singularity**; replace the small jumps by an artificial diffusion term.
- R_z^{II} : **near the singularity**; use singularity-aware quadrature weights.
- R_z^{III} : **away from the singularity**; ordinary midpoint quadrature is adequate.

Let

$$f(z, x, t) = u(x + \gamma(x, z), t) - u(x, t) - \gamma(x, z)^\top \nabla_x u(x, t).$$

Quadrature formula over a given cell (midpoint z_n):

► R_z^{I} , at the singularity:

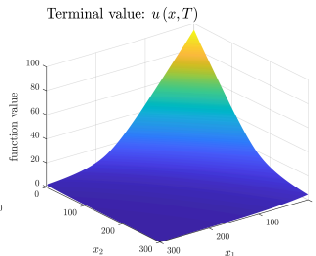
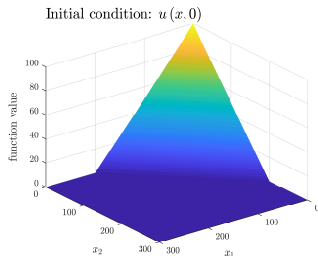
$$\int_{\text{cell}} f(z, x, t) \ell(z) dz \approx \sum_{i,j=1}^2 u_{xx}^{(i,j)}(x, t) \int_{\text{cell}} z_i z_j \ell(z) dz$$

► R_z^{II} , close to the singularity:

$$\int_{\text{cell}} f(z, x, t) \ell(z) dz \approx \frac{f(z_n, x, t)}{\|z_n\|^2} \int_{\text{cell}} \|z\|^2 \ell(z) dz$$

► R_z^{III} , away from the singularity:

$$\int_{\text{cell}} f(z, x, t) \ell(z) dz \approx f(z_n, x, t) \ell(z_n) \int_{\text{cell}} dz$$



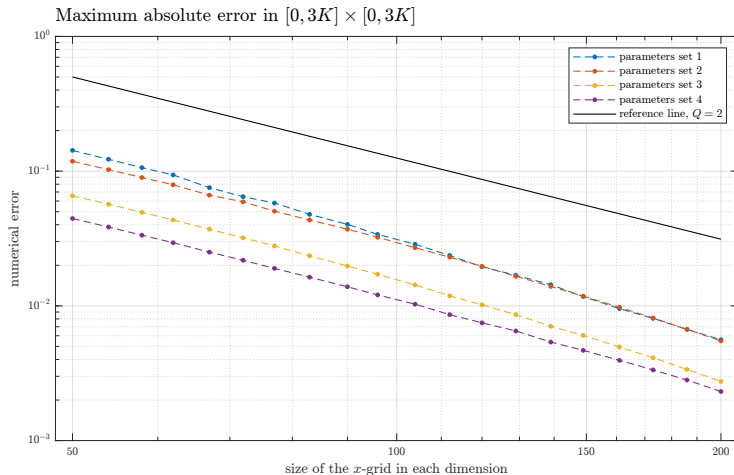
The payoff is

$$\phi(x) = \max\left(K - \frac{x_1 + x_2}{2}, 0\right).$$

The two surfaces show the initial payoff and the option value after time evolution.

Test family

Four representative NTS parameter sets: two VG and two NIG cases.



The error is the maximum absolute error on $[0, 3K]^2$.

- ▶ All four test sets exhibit a favourable rate close to two.
- ▶ This matches the theorem for the VG sets.
- ▶ It is **better than the available bound** for the NIG sets.

This agrees with results for the 1D case of Wang *et al.* (2007), i.e.

$$\text{error} = O\left(M^{-\min(2, 3-2\alpha)}\right)$$

while our consistency analysis gives the rate

$$\text{error} = O\left(M^{-(2-2\alpha)}\right).$$

At present, we do not have a clear explanation for the observed second-order convergence in the two NIG sets, which we leave for future work.

Our main results are:

- ▶ we provide a **rigorous proof of the convergence behaviour** of the proposed numerical quadrature scheme;
- ▶ we perform **numerical experiments** to assess the accuracy of the proposed scheme.

Swing options

M. Regragui, K.J. in 't Hout, M. Vanmaele, F.E. Benth,
Numerical methods for solving PIDEs arising in swing option pricing under a two-factor mean-reverting model with jumps, Applied Numerical Mathematics (2026).

Infinite-activity two-asset options

M. Moda, K.J. in 't Hout, M. Vanmaele, F.E. Benth,
Numerical Valuation of European Options Under Two-Asset Infinite-Activity Exponential Lévy Models, Applied Mathematical Finance (2026).

See the posters by M. Moda and M. Regragui for extensions of this work.

Thank you for your attention.

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