

Neural Field Equations with Random Input

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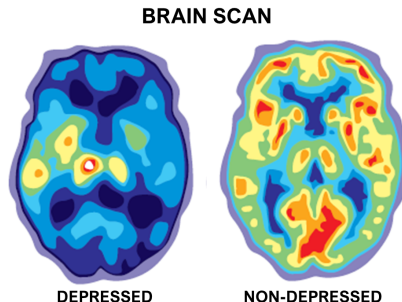
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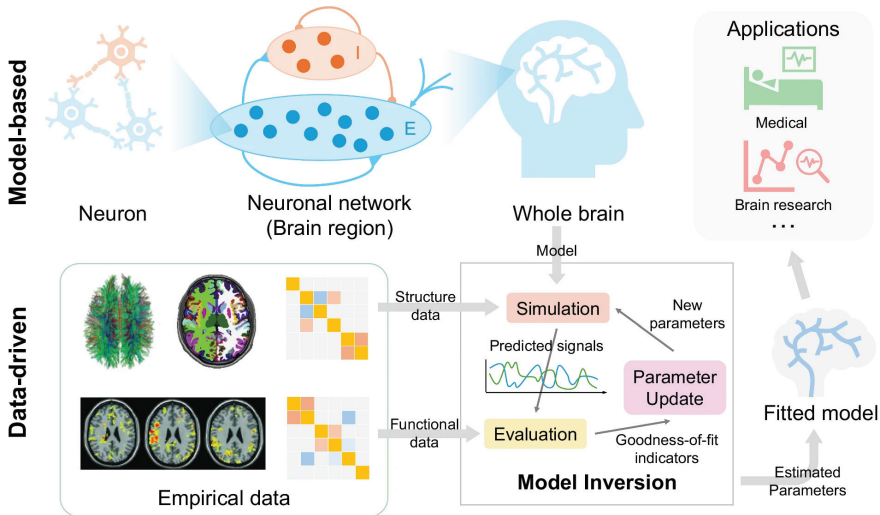
Why Model Brain Dynamics?

- Depression affects around 322 million people worldwide – about 1 in 20 adults – and is a leading cause of disability.
- Even in high-income countries, only about a third of people with depression receive treatment.
- Personalized whole-brain models are already used to plan deep brain stimulation for treatment-resistant depression.
- Model parameters are never known exactly, so quantifying uncertainty is essential for clinical trust.



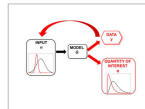
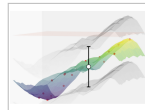
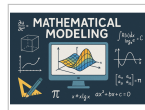
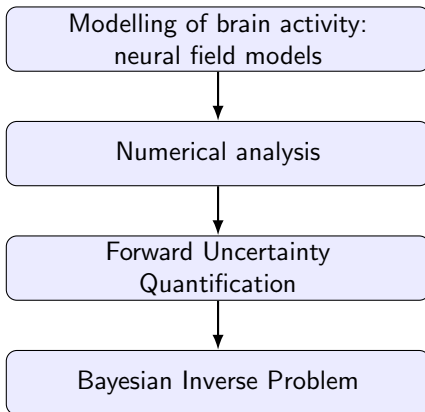
From USA Brain Sciences

Model-based and data-driven approaches



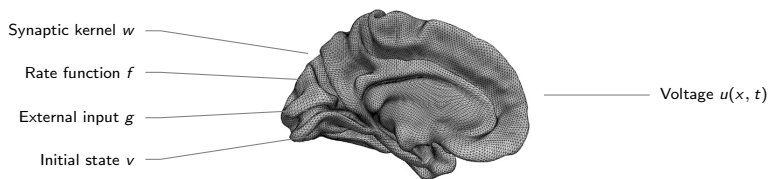
Ref: Zheng, Z., Wei, J., Xu, Y. et al. (2025) Modeling macroscopic brain dynamics with brain-inspired computing architecture. Nat Commun.

Rigorous mathematics is lacking



⇒ This talk develops missing mathematical foundations for neural field models in terms of Uncertainty Quantification and Inverse Problem.

Neural Field Equations



$$\partial_t u(x, t) = -u(x, t) + \int_D w(x, x') f(u(x', t)) dx' + g(x, t), \quad (x, t) \in D \times J$$
$$u(x, 0) = v(x), \quad x \in D.$$

Ref: Wilson and Cowan (1973), A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue;
Amari (1977) Dynamics of pattern formation in lateral-inhibition type neural fields.

Ref: Faugeras et al (2009) Persistent Neural States: Stationary Localized Activity Patterns in Nonlinear Continuous
 n -Population, q -Dimensional Neural Networks; Potthast and Graben (2010) Existence and properties of solutions for neural field equations.

Mesh: Lila, Aston & Sangalli (2016), Smooth PCA over two-dimensional manifolds with an application to neuroimaging, Ann. Appl. Statist.

Continuous Neural Field Equations with Random Input: Well-Posedness

Theorem (Avitabile, Cavallini, D., and Lord (2026) Neural fields equations with random data, SIAM UQ)

Under general conditions on w , f , and g , if the initial condition $v \in L^p(\Omega, C(D))$, the neural field equation has a unique solution u that is continuously differentiable on a finite time interval for $\omega \in \Omega$, $\mathbb{P}$ -almost surely, and u is in the space of p -integrable functions on D .

Equivalent results hold for random w , f , and g .

Theorem (Avitabile, Cavallini, D., and Lord (2026) Stochastic collocation schemes for Neural Field Equations with random data)

For a linear neural field model under general assumptions on w , f , and g , and a finite-dimensional random assumption on v , if the initial condition v_n satisfies a regularity assumption, then the finite-dimensional solution u_n is well-posed and it is regular.

Equivalent results hold for finite-dimensional random w , f , and g .

Numerical example

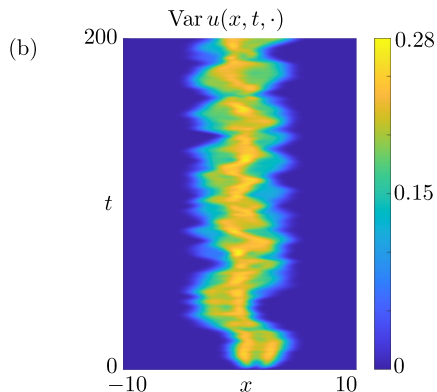
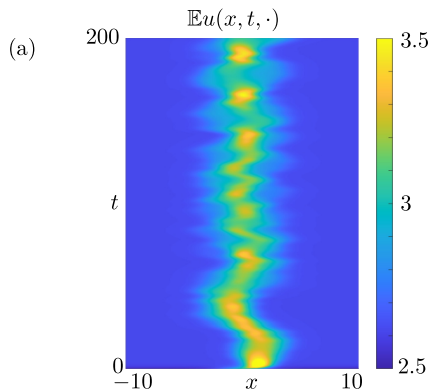
We consider a linear neural field on one-dimensional space $D = [-L, L]$ and with $J = [0, T]$.

$$\partial_t u(x, t) = -u(x, t) + \int_D x x' u(x', t) dx' + g(x, t), \quad (x, t) \in D \times J$$

$$u(x, 0) = \sin(4\pi x), \quad x \in D.$$

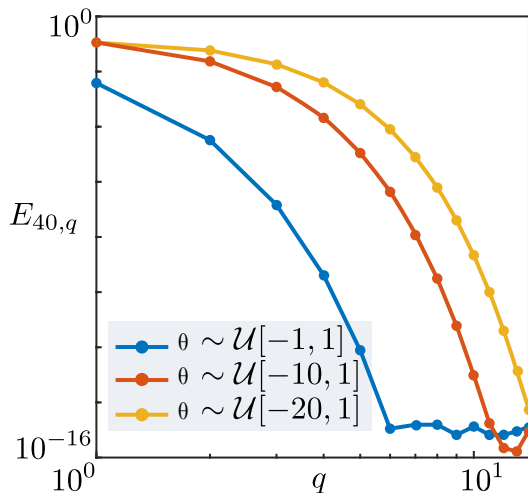
- Uncertainty is in the external forcing $g(x, t, \theta)$ where θ is a finite dimensional uncertain parameter uniformly distributed.
- Space discretisation (in n): Chebyshev spectral collocation paired with Clenshaw-Curtis quadrature. $\Rightarrow$ This gives spectral convergence for a fixed parameter θ .
- Time discretisation: 4th-order explicit adaptive Runge-Kutta. $\Rightarrow$ We set tolerance to 10^{-15} .
- Parameter discretisation (in q): Stochastic Collocation with Gauss-Legendre quadrature.

Expectation and variance of a solution u to the neural field



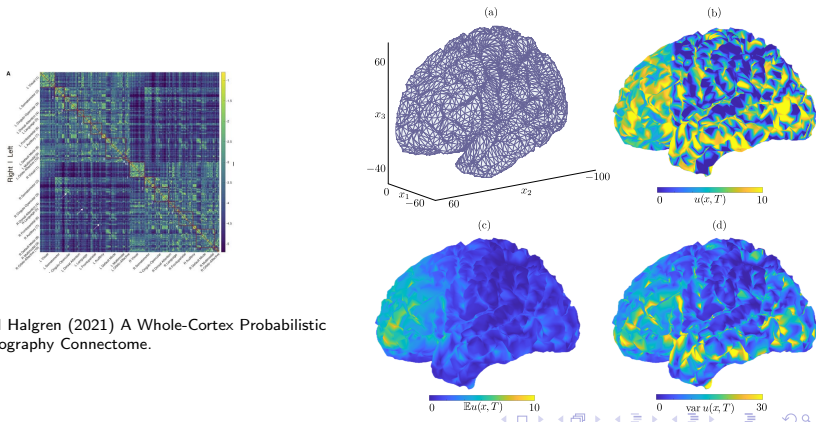
Stochastic Collocation method: convergence

For the linear neural field model, we showed that regular schemes such as Stochastic Collocation methods give exponential convergence.



Example of a forward UQ problem for the neural field model

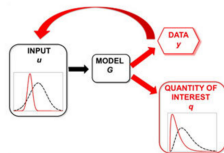
Synaptic kernel on the left and the corresponding solution on the right: the block structure visible in the matrix w reflects strong connectivity within cortical regions and hemispheres.



Ref: Rosen and Halgren (2021) A Whole-Cortex Probabilistic Diffusion Tractography Connectome.

Bayesian Inverse Problem: continuous and discrete

Given finite dimensional partial data and prior on uncertain input (for example initial condition), the Bayesian Inverse Problem aims to provide the posterior conditioned on the data.



Bayesian Inverse Problem

We seek to recover the unknown input θ from noisy observations

$$y = G(\theta) + \eta,$$

where θ is the unknown input we wish to infer, $G(\theta)$ is the forward model (e.g. the neural field solution operator), y is the observed data, and $\eta \sim \mathbb{Q}_0$ is the observational noise with $\mathbb{Q}_0 = \mathcal{N}(0, \Sigma)$.

Bayesian approach: place a prior μ_0 on θ ; Bayes' rule gives the posterior

$$\frac{d\mu^y}{d\mu_0}(\theta) \propto \exp(-\Phi(\theta; y)), \quad \Phi(\theta; y) = \frac{1}{2} \|\Sigma^{-1/2}(y - G(\theta))\|^2.$$

Goal: characterize the full posterior μ^y , not just a point estimate.

Ref: Stuart (2010), Inverse problems: a Bayesian perspective, Acta Numerica.

Inverse Problem Formulation

Let $H : C(J, \mathbb{X}) \rightarrow \mathbb{Y}$ be a Lipschitz observable quantity, and define the parameter-to-observation map

$$G(\theta) = H(u(\cdot, \theta)) = (H \circ \Psi)(\theta), \quad \theta \in \mathbb{D},$$

where $\Psi : \mathbb{D} \rightarrow C(J, \mathbb{X})$ maps parameters to neural field solutions. We consider noisy, partial observations

$$y = G(\theta) + \eta, \quad \theta \sim \mu_0 \text{ (prior)}, \quad \eta \sim \mathbb{Q}_0 \text{ (noise)}, \quad \eta \perp \theta,$$

with likelihood given by the Radon–Nikodym derivative

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{Q}_0}(y) = \exp(-\Phi(\theta, y)), \text{ where } \Phi : \mathbb{D} \times \mathbb{Y} \rightarrow \mathbb{R} \text{ is the } \textit{potential}.$$

Generalized Bayes' theorem: if $Z(y) := \int_{\mathbb{D}} \exp(-\Phi(\theta, y)) \mu_0(d\theta) > 0$ for $\mathbb{Q}_0$ -a.e. y , the posterior μ^y of $\theta \mid y$ exists, $\mu^y \ll \mu_0$, and

$$\frac{d\mu^y}{d\mu_0}(\theta) = \frac{1}{Z(y)} \exp(-\Phi(\theta, y)).$$

Inferring the Initial Condition: Setup

Fix (w, f, g) and set $\theta = v$, so $\mathbb{D} = \mathbb{X} \hookrightarrow \mathbb{H} = L^2(D)$.

- **Spatial projector:** $P_n \in BL(\mathbb{X}, \mathbb{X}_n)$, $\mathbb{X}_n = \text{Span}\{\varphi_1, \dots, \varphi_{s(n)}\}$;
- **Prior projector:** the orthogonal $Q_m^\vee \in BL(\mathbb{H}, \mathbb{H}_m)$,
 $\mathbb{H}_m = \text{Span}\{e_1, \dots, e_{r(m)}\} \subset \mathbb{X}$,

$$(Q_m^\vee v)(x) = \sum_{k \in \mathbb{N}_{r(m)}} \langle v, e_k \rangle_{L^2(D)} e_k(x).$$

Continuous forward problem:

$$u'(t) = -u(t) + WF(u(t)) + g(t), \quad t \in J, \quad u(0) = v,$$

and the spatially-discretised problem with a *perturbed* (discretised) initial condition $Q_m^\vee v$:

$$u'_{n,m}(t) = -u_{n,m}(t) + P_n WF(u_{n,m}(t)) + P_n g(t), \quad u_{n,m}(0) = P_n Q_m^\vee v.$$

Goal: bound the combined error $\|v - P_n Q_m^\vee v\|_{\mathbb{X}}$ as $n, m \rightarrow \infty$.

Inferring the Initial Condition: Combined Error Bound

Theorem (Avitabile, Dashti, D, in progress)

Under general assumptions and assuming uniformly bounded spatial projectors, if $e_k \in \mathbb{X}^\alpha$ for all $k \in \mathbb{N}$, and there exists a bounded $\delta : \mathbb{N} \rightarrow \mathbb{R}_{>0}$ and a bounded $\gamma_\alpha : \mathbb{N} \rightarrow \mathbb{R}_{>0}$ with

$$\|v - Q_m^\vee v\|_{\mathbb{X}} \leq \delta(m) \|v\|_{\mathbb{X}^\alpha}, \quad v \in \mathbb{X}^\alpha, \quad m \in \mathbb{N},$$

and

$$\|v - P_n v\|_{\mathbb{X}} \leq \gamma_\alpha(n) \|v\|_{\mathbb{X}^\alpha}, \quad v \in \mathbb{X}^\alpha, \quad n \in \mathbb{N},$$

then

$$\|v - P_n Q_m^\vee v\|_{\mathbb{X}} \lesssim [\gamma_\alpha(n) + \delta(m)] \|v\|_{\mathbb{X}^\alpha}, \quad n, m \in \mathbb{N}.$$

Theorem (Avitabile, Dashti, D, in progress)

For the continuous neural field model, under the assumptions of absolute continuity and finite-dimensional data, the posterior measure is well defined. For the semi-discrete neural field model, an equivalent result holds.

Numerical example

We consider a linear neural field on one-dimensional space $D = [-L, L]$ and with $J = [0, T]$.

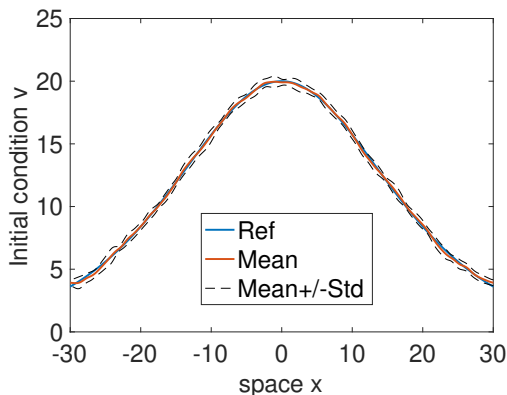
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$$u(x, 0) = v(x), \quad x \in D.$$

- Uncertainty is in the initial condition $v(x, \theta)$ where θ is a finite dimensional uncertain parameter normally distributed.
- Space discretisation: Chebyshev spectral collocation paired with Clenshaw-Curtis quadrature. $\Rightarrow$ This gives spectral convergence for a fixed parameter y .
- Time discretisation: 4th-order explicit adaptive Runge-Kutta. $\Rightarrow$ We set tolerance to 10^{-14} .
- Prior: Karhunen-Loeve expansion.
- Posterior: approximated with an MCMC method.

Example of an Inverse Problem for the neural field model

Given a Karhunen-Loeve expansion with uncertain coefficients as the prior for the IC and finite-dimensional partial noisy data of the solution over space and time, we were able to obtain an accurate approximation of the reference IC.



Conclusions

- We showed well-posedness of neural field model for both non-parametric and parametric random input.
- We derived error convergence results for the linear neural field model.
- We showed well-posedness of the Bayesian Inverse Problem for continuous and semi-discrete neural field models.