

Monolithic solvers for parabolic PDEs

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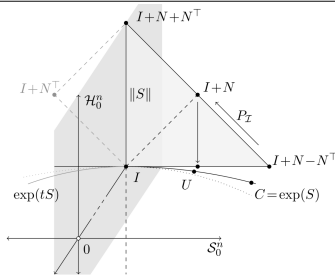
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Chris Stolk – my research



- ▶ Inverse problems for wave equations
- ▶ Time-harmonic wave equations (multigrid, domain decomposition, time-domain solvers)
- ▶ Various topics in modeling and PDEs

Jan Brandts, Korteweg-de Vries Instituut, UvA



Copositive
Programming
NP-hard Cone
Membership
Nonnegative
Factorization



MasterMath's NLA:

GMRES | CG | IRA
Weyl | Cato-Temple
Faber-Manteuffel
Random Sketching
Johnson-Lindenstrauss

Jan Brandts
Sergey Korotov
Michal Krížek

Simplicial Partitions
with Applications
to the Finite
Element Method

 Springer

Outline

Ultimate aim: (Provably) optimal adaptive solvers —known for elliptic problems— for time-dependent problems.

- 1 Motivation: Recap of results for elliptic problems
- 2 Time-marching schemes for parabolic problems
- 3 Simultaneous space-time variational formulations and minimal residual methods
 - ‘Canonical’ spaces
 - Fractional spaces
 - First order system
- 4 Nonlinear spatial PDOs

Adaptivity for elliptic problems I

- Variational formulation: Given $f \in V'$, $a: V \times V \rightarrow \mathbb{R}$ bilinear, bounded, **coercive**, search $u \in V$

$$a(u, v) = f(v) \quad (v \in V) \iff Au = f,$$

with $(Au)(v) := a(u, v)$ (e.g. $\int_{\Omega} \nabla u \cdot \nabla v \, dx$).

- $\frac{1}{C} \|f\|_{V'} \leq \|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'}$. **Well-posedness.**
- Galerkin: $u_h \in V_h \subset V$,

$$a(u_h, v_h) = f(v_h) \quad (v_h \in V_h).$$

Cea's lemma:

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V. \quad \text{Quasi-best approximation.}$$

Adaptivity for elliptic problems II

- For $\Omega \subset \mathbb{R}^d$, let $V = H_0^1(\Omega)$. V_h f.e. space of degree p w.r.t. quasi-uniform mesh, mesh-size h .

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \lesssim h^p \|u\|_{H^{p+1}(\Omega)}. \quad N := \dim V_h \approx h^{-d}$$

$$\overset{\text{Cea}}{\rightsquigarrow} \|u - u_h\|_{H^1(\Omega)} \lesssim N^{-\frac{p}{d}} \|u\|_{H^{p+1}(\Omega)}.$$

- $u \in H^{p+1}(\Omega)$ requires $\partial\Omega$ and f to be smooth.
- For $d = 2$, and Ω with re-entrant corner of angle $\frac{\pi}{\beta}$ with $\beta \in [\frac{1}{2}, 1)$, gen. at best $u \in H^{1+\beta-\varepsilon}(\Omega)$. With quasi-unif. meshes, rate $\frac{p}{d}$ reduces to $\frac{\beta}{d}$.
- For smooth f , **maximal** rate $\frac{p}{d}$ can be retrieved by **local refinement** of mesh towards corner (Dahlke & DeVore '97) (not true for $d > 2$)
- Such a (seq. of) mesh(es) can be found by AFEM (SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE) using a suitable **a posteriori error estimator**:

Theorem 1

Whenever u can be **approximated** by f.e. functions at rate $s \in (0, \frac{p}{d}]$, Galerkin solutions by AFEM converge with this (**optimal**) rate s (in linear complexity).

Time-marching schemes for parabolic problems I

$$\begin{cases} \frac{du}{dt}(t) + A(t)u(t) &= \ell(t) & t \in J := (0, T), \\ u(0) &= u_0, \end{cases}$$

$A(t) \in \mathcal{L}(V, V')$ coercive (e.g. $V = H_0^1(\Omega)$, $A(t) \equiv A$, $(Au)(v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$).

- *Semi-discr.-in-space*: $V_h \subset V$, say cont. piecewise lin., search $u_h(t) \in V_h$,

$$\begin{cases} \langle \frac{du_h}{dt}(t), v_h \rangle_{L_2(\Omega)} + (A(t)u_h(t))(v_h) &= \langle \ell(t), v_h \rangle_{L_2(\Omega)} & v_h \in V_h, t \in J, \\ \langle u_h(0), v_h \rangle_{L_2(\Omega)} &= \langle u_0, v_h \rangle_{L_2(\Omega)} & v_h \in V_h. \end{cases}$$

- *Finite-difference-in-time* (here Crank-Nicolson). Time step k , $t_n := nk$, for $n = 1, 2, \dots$, approx. $u(t_n)$ by $U^n \in V_h$ from

$$\begin{cases} \langle \frac{U^n - U^{n-1}}{k}, v_h \rangle_{L_2(\Omega)} + (A(t) \frac{1}{2} (U^n + U^{n-1}))(v_h) &= \langle \ell(\frac{t_n + t_{n-1}}{2}), v_h \rangle_{L_2(\Omega)} & v_h \in V_h, \\ \langle U^0, v_h \rangle_{L_2(\Omega)} &= \langle u_0, v_h \rangle_{L_2(\Omega)} & v_h \in V_h. \end{cases}$$

Time-marching schemes for parabolic problems II

Theorem 2

Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ (compatibility). Then

$$\begin{aligned} \|U^n - u(t_n)\|_{L_2(\Omega)} &\lesssim h^2 (\|u_0\|_{H^2(\Omega)} + \int_0^{t_n} \|\frac{du}{dt}\|_{H^2(\Omega)} dt) \\ &\quad + k^2 (\int_0^{t_n} \|\frac{d^3u}{dt^3}\|_{L_2(\Omega)} + \|\Delta_x \frac{d^2u}{dt^2}\|_{L_2(\Omega)} dt). \end{aligned}$$

(Thomée '96)

- So for $h \approx k$ maximal rate, but under (severe) regularity conditions only.
- No quasi-best approximation result.

'Canonical' space-time variational formulation

$$\begin{cases} \frac{du}{dt}(t) + A(t)u(t) &= \ell(t) & t \in J := (0, T), \\ u(0) &= u_0, \end{cases}$$

$A(t) \in \mathcal{L}(V, V')$ coercive, $\ell(t) \in V'$, $u_0 \in H$, where $V \hookrightarrow H \simeq H' \hookrightarrow V'$.

$$\underbrace{\iint_{J \times \Omega} \frac{du}{dt}(t, x) v(t, x) dt dx}_{(d_t u)(v) :=} + \underbrace{\int_J \overbrace{(A(t)(u(t))(v(t)))}^{\text{e.g. } \int_{\Omega} \nabla_x u(t, x) \cdot \nabla_x v(t, x) dx} dt}_{(Au)(v) :=} = \underbrace{\iint_{J \times \Omega} \ell(t, x) v(t, x) dt dx}_{(\ell u)(v) :=}$$

- With $X := L_2(J; V) \cap H^1(J; V')$, $Y := L_2(J; V)$, $B := d_t + \mathcal{A} \in L(X, Y')$.
- ▶ $X \hookrightarrow C(\bar{J}; H)$, i.e. $\gamma_t := w \mapsto w(t, \cdot) \in \mathcal{L}(X, H)$.
- ▶ With $B_e := (B, \gamma_0)^\top$, solve $B_e u = (\ell, u_0)^\top$.

Theorem 3 (Well-posedness)

$B_e \in \mathcal{L}is(X, Y' \times H)$ (Lions & Magenes '72)

$$\text{▶ } \int_J \frac{du}{dt} v + u \frac{dv}{dt} dt = \int_J \frac{d(uv)}{dt} dt = u(T)v(T) - u(0)v(0) \rightsquigarrow d_t + d'_t = \gamma'_T \gamma_T - \gamma'_0 \gamma_0.$$

Minimal residual method I

- $B_e u = (\ell, u_0)^\top$ well-posed, but $Y' \times H \neq X' \rightsquigarrow$ no Galerkin.
- $u = \operatorname{argmin}_{w \in X} \|\ell - Bw\|_{Y'}^2 + \|u_0 - \gamma_0 w\|_H^2$.
- (For simpl.) let $\mathcal{A} = \mathcal{A}'$, equip $Y = L_2(J; V)$ with $\|\cdot\|_Y := (\mathcal{A}\cdot)(\cdot)^{\frac{1}{2}}$.
- Eq. for uniq. stat. pt.

$$(B' \mathcal{A}^{-1} B + \gamma'_0 \gamma_0) u = B' \mathcal{A}^{-1} \ell + \gamma'_0 u_0.$$

- With $\lambda := \mathcal{A}^{-1}(\ell - Bu) \in Y$ ($= 0$), equiv. saddle-point system

$$\begin{bmatrix} \mathcal{A} & B \\ B' & -\gamma'_0 \gamma_0 \end{bmatrix} \begin{bmatrix} \lambda \\ u \end{bmatrix} = \begin{bmatrix} \ell \\ -\gamma'_0 u_0 \end{bmatrix}.$$

Operator at lhs $\in \mathcal{L}(\mathcal{Y} \times X, Y' \times X')$.

- Families $(X^\delta)_{\delta \in \Delta} \subset X$, $(Y^\delta)_{\delta \in \Delta} \subset Y$ of 'trial'- and 'test'-spaces. Denoting inclusions $X^\delta \subset X$, $Y^\delta \subset Y$ by E_X^δ and E_Y^δ , Galerkin systems

$$\begin{bmatrix} \mathcal{A}^\delta & B^\delta \\ (B^\delta)' & -(\gamma_0^\delta)' \gamma_0^\delta \end{bmatrix} \begin{bmatrix} \lambda^\delta \\ u^\delta \end{bmatrix} = \begin{bmatrix} (E_Y^\delta)' \ell \\ -(\gamma_0^\delta)' u_0 \end{bmatrix}. \quad (\text{MR}^\delta)$$

$$\mathcal{A}^\delta := (E_Y^\delta)' \mathcal{A} E_Y^\delta, \quad B^\delta := (E_Y^\delta)' B E_X^\delta, \quad \gamma_0^\delta := \gamma_0 E_X^\delta.$$

Minimal residual method II

- For any $X^\delta \subset Y^\delta$ (MR^δ) is inf-sup stable. Needed: *uniform* inf-sup stability.

Theorem 4 (Quasi-best approx.)

Set

$$\gamma_\delta := \inf_{\{w \in X^\delta: d_t w \neq 0\}} \frac{\sup_{y \in Y^\delta} \frac{(d_t w)(y)}{\|y\|_Y}}{\|d_t w\|_{Y'}},$$

If $\gamma_\Delta := \inf_{\delta \in \Delta} \gamma_\delta > 0$, then

$$\|u - E_X^\delta u^\delta\|_X \leq \frac{1}{\gamma_\Delta} \inf_{w \in X^\delta} \|u - E_X^\delta w\|_X.$$

The uniform inf-sup condition $\gamma_\Delta > 0$ I

Lemma 5 (tensor product setting, Andreev 2013)

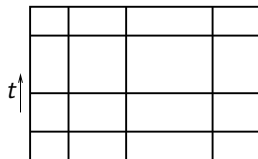
Let $X^\delta = X_t^\delta \otimes X_x^\delta$, $Y^\delta = Y_t^\delta \otimes Y_x^\delta$. Then $\gamma_\Delta \geq \gamma_{\Delta,t} \gamma_{\Delta,x}$, where

$$\gamma_{\Delta,t} := \inf_{\delta \in \Delta} \inf_{\{z \in X_t^\delta : z' \neq 0\}} \frac{\sup_{0 \neq v \in Y_t^\delta} \frac{\int_J z' v \, dt}{\|v\|_{L_2(J)}}}{\|z'\|_{L_2(J)}}, \quad \gamma_{\Delta,x} := \inf_{\delta \in \Delta} \inf_{0 \neq z \in X_x^\delta} \frac{\sup_{0 \neq v \in Y_x^\delta} \frac{\langle z, v \rangle_H}{\|v\|_V}}{\|z\|_{V'}}.$$

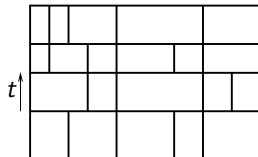
- ▶ $\gamma_{\Delta,t} = 1$ when $dX_t^\delta \subset Y_t^\delta$.
- ▶ For $Y_x^\delta = X_x^\delta$, $\gamma_{\Delta,x} = \inf_{\delta \in \Delta} \|Q^\delta\|_{\mathcal{L}(V,V)}^{-1}$ with Q^δ H -orth. proj. onto X_x^δ .
- ▶ For $(H, V) = (L_2(\Omega), H_0^1(\Omega))$, $\Omega \subset \mathbb{R}^d$ and $d < 7$, $\gamma_{\Delta,x} > 0$ for the families of Lagrange finite element spaces of arbitrary degree w.r.t. all conforming simplicial meshes that can be created by newest vertex bisection from an initial mesh (Diening, Storn & Tscherpel 2025).

The uniform inf-sup condition $\gamma_{\Delta} > 0$ II

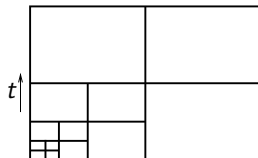
Unif. inf-sup for **tensor product setting**



generalizes to **time-slab setting**



but *not* to partitions with 'local time-steps'
(at least not for $\dim Y^{\delta} \lesssim \dim X^{\delta}$)



(In d -dims, prismatic partitions, i.e., intervals-in-time $\times$ d -simplices-in-space)

A data-dependent condition for quasi-best approx.

Unif. inf-sup gives quasi-best approx. *uniformly in all data* $(\ell, u_0) \in Y' \times H$.
Inspired by Cohen, Dahmen & Welper 2012, here a condition for *data at hand*:

Theorem 6

Given $(\ell, u_0) \in Y' \times H$, let $(\lambda^\delta, u^\delta) \in Y^\delta \times X^\delta$ be the solution of (MR $^\delta$).
With $\lambda_\delta := \mathcal{A}^{-1}(\ell - Bu^\delta)$, for some $\varrho \geq 0$ let

$$\|\lambda_\delta - \lambda^\delta\|_Y \leq \varrho \sqrt{\|\lambda^\delta\|_{Y^\delta}^2 + (1 + \frac{1}{(\varrho + \frac{1}{2})^2}) \|u_0 - \gamma_0 u^\delta\|_H^2}. \quad (\text{apost}^\delta)$$

Then

$$\|u - u^\delta\|_X \leq \sqrt{\varrho^2 + \frac{9\varrho+4}{4\varrho+4}} \inf_{\bar{u}^\delta \in X^\delta} \|u - \bar{u}^\delta\|_X.$$

Moreover, with $\eta(\delta)^2 := \|\lambda^\delta\|_{Y^\delta}^2 + \|u_0 - \gamma_0 u^\delta\|_H^2$,

$$\eta(\delta)^2 \leq \|u - u^\delta\|_X^2 \leq (1 + \varrho^2(1 + \frac{1}{(\varrho + \frac{1}{2})^2})) \eta(\delta)^2.$$

Double adaptivity

For family $(\mathcal{P}^\delta)_{\delta \in \Delta}$ prismatic partitions of $J \times \Omega$, and $p \in \mathbb{N}$,

$$X^\delta := X \cap Q_{p,p}(\mathcal{P}^\delta),$$

and for some $\mathcal{P}^\underline{\delta} \succeq \mathcal{P}^\delta$, $Y^\delta := Y \cap Q_{p,p}(\mathcal{P}^\underline{\delta})$.

double-adaptive solver (ℓ, u_0)

$\delta = \underline{\delta} := \delta_0$. Select constant $\varrho > 0$.

(I) **solve** $(\lambda^\delta, u^\delta) \in Y^\delta \times X^\delta$ from (MR^δ) .

if $\neg (\text{apost}^\delta)$

then enrich $\mathcal{P}^\underline{\delta}$ by **mark & refine** on (a post. err. est. for) $\|\lambda_\delta - \lambda^\delta\|_Y^2$.

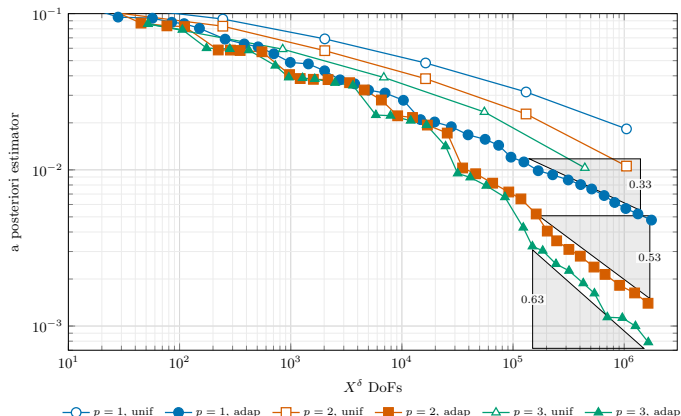
goto (I)

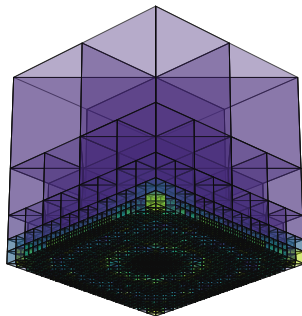
else enrich $\mathcal{P}^\delta$ by **mark & refine** on $\eta(\delta)^2$. $\mathcal{P}^\underline{\delta} := \mathcal{P}^\delta$, goto (I).

endif

Numerics

Heat eq. $\Omega = (0, 1)^2$. $\ell = 0$, $u_0(x, y) = \sqrt{(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2} \sin \pi x \sin \pi y$.
For $p \in \{1, 2, 3\}$, $X^\delta = X \cap Q_{p,p}(\mathcal{P}^\delta)$, $Y^\delta = Y \cap Q_{p,p}(\mathcal{P}^\delta)$. **No** inner refinements were made, so $\mathcal{P}^\delta = \mathcal{P}^\delta$. Maximal rate $p/3$.





- Direct solver.
- So far optimal preconditioner only for tensor product case. Superior parallel complexity compared to Parareal (Westerdiep & van Venetië 2022).
- Monolithic methods require more memory. Disadvantage vanishes when whole time evolution is needed simultaneously, as with optimal control.

Other space-time formulations: Fractional spaces

- $B = d_t + \mathcal{A}$, $B_e = (B, \gamma_0)^\top \in \mathcal{L}\text{is}\left(L_2(J; V) \cap H^1(J; V), L_2(J; V)' \times H\right)$.
- For hom. init. cond., with $H_{0,}^1(J) := \{v \in H^1(J) : v(0) = 0\}$,

$$B \in \mathcal{L}\text{is}\left(L_2(J; V) \cap H_{0,}^1(J; V'), L_2(J; V')'\right).$$

- Int-by-parts over time shows, with $H_{,0}^1(J) := \{v \in H^1(J) : v(T) = 0\}$,

$$B \in \mathcal{L}\text{is}\left(L_2(J; V), (L_2(J; V) \cap H_{,0}^1(J; V'))'\right).$$

- Real interpolation shows*

$$B \in \mathcal{L}\text{is}\left(L_2(J; V) \cap H_{0,}^{1/2}(J; H), (L_2(J; V) \cap H_{,0}^{1/2}(J; H))'\right).$$

- For arb. init. cond. $u_0 \in H$,

$$B_e \in \mathcal{L}\text{is}\left(\overbrace{L_2(J; V) \cap H_{0,}^{1/2}(J; H) + L_2(J; V) \cap H^1(J; V)}^{\hookrightarrow L_2(J; V) \cap H^{1/2}(J; H)}, (L_2(J; V) \cap H_{,0}^{1/2}(J; H))' \times H\right).$$

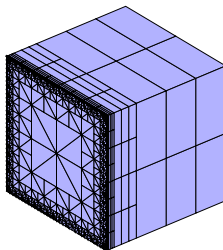
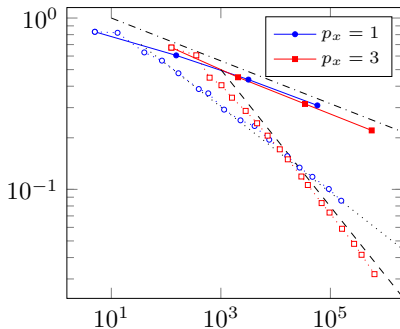
*Fontes 2009, St. & Schwab 2017, Steinbach & Zank 2020

Fractional spaces: Num. approx.

- Minimal residual method $\rightsquigarrow$ saddle-point.
- With $H := L_2(\Omega)$, $V := H_0^1(\Omega)$, unif. inf-sup stability for trial and test spaces being f.e. spaces w.r.t. *same, general* prismatic partition, assuming that prisms P are *parabolically scaled*, i.e. $P = I_P \times T_P$ with $\text{diam } I_P \approx \text{diam } T_P^2$.
- (Near)-optimal preconditioner.
- Roughly speaking: parabolic scaling compensates for anisotropic character $L_2(J; H_0^1(\Omega)) \cap H^{1/2}(J; L_2(\Omega))$

Fractional spaces: Numerics

Heat eq. $\Omega = (0, 1)^2$, $X^\delta = C(J \times \Omega) \cap L_2(J; H_0^1(\Omega)) \cap Q_{p_t, p_x}(\mathcal{P}^\delta)$, $p_t = 1$, $p_x \in \{1, 3\}$. Maximal rates $\{\frac{1}{4}, \frac{3}{4}\}$.
Data $\ell = 0$, $u_0 = 1$.



Dashed-dotted, dotted, dashed corr. to $\{\frac{1}{8}, \frac{1}{4}, \frac{2}{5}\}$.

Other space-time formulations: FOSLS I

For $(\ell, u_0) \in L_2(J; H^{-1}(\Omega)) \times L_2(\Omega)$,

$$\begin{cases} \frac{du}{dt} - \Delta_x u = \ell & J \times \Omega^\dagger \\ u = 0 & J \times \partial\Omega \\ u(0, \cdot) = u_0 & \Omega \end{cases}$$

Write $\ell = \ell_1 + \operatorname{div}_x \vec{\ell}_2$ for $(\ell_1, \vec{\ell}_2) \in L_2(\Omega) \times L_2(\Omega)^d$.

Then $\vec{u} = (u_1, \vec{u}_2) := (u, -\nabla_x u - \vec{\ell}_2)$ satisf.

$$G\vec{u} := \begin{pmatrix} \frac{du_1}{dt} + \operatorname{div}_x \vec{u}_2 \\ -\vec{u}_2 - \nabla_x u_1 \\ u_1(0, \cdot) \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \vec{\ell}_2 \\ u_0 \end{pmatrix}, \quad u_1|_{J \times \partial\Omega} = 0.$$

Theorem 7 (Führer & Karkulik 2021, Gantner & St. 2021)

For $U := \{\vec{u} = (u_1, \vec{u}_2) \in L_2(J; H_0^1(\Omega)) \times L_2(J \times \Omega)^d : \underbrace{\operatorname{div} \vec{u}}_{\frac{du_1}{dt} + \operatorname{div}_x \vec{u}_2} \in L_2(J \times \Omega)\}$

with graph-norm, $G \in \mathcal{L}is(U, L_2(J \times \Omega) \times L_2(J \times \Omega)^d \times L_2(\Omega))$.

[†]gener. of $-\Delta_x$ possible

Other space-time formulations: FOSLS II

Corollary 8 (quasi-best approx.)

For *any* finite dimensional $U^\delta \subset U$, and

$$\vec{u}^\delta := \operatorname{argmin}_{\vec{w} \in U^\delta} \|G\vec{w} - (\ell_1, \vec{\ell}_2, u_0)\|_{L_2(J \times \Omega) \times L_2(J \times \Omega)^d \times L_2(\Omega)}^2,$$

$$\|\vec{u} - \vec{u}^\delta\|_U \leq \|G\| \|G^{-1}\| \inf_{\vec{w} \in U^\delta} \|\vec{u} - \vec{w}\|_U.$$

Moreover,

$$\|\vec{u} - \vec{u}^\delta\|_U \approx \|(\ell_1, \vec{\ell}_2, u_0) - G\vec{u}^\delta\|_{L_2(J \times \Omega) \times L_2(J \times \Omega)^d \times L_2(\Omega)}.$$

- Construction of **optimal preconditioner** WIP.
- Because of presence $\|\operatorname{div} \bullet\|_{L_2(J \times \Omega)}$ in graph-norm, norm on U is stronger than those in previous two formulations.

Nonlinear spatial operators

- $V \hookrightarrow H \simeq H' \hookrightarrow V'$, $A(t): V \rightarrow V'$, $\ell(t) \in V'$, $u_0 \in H$,

$$\begin{cases} \frac{du}{dt}(t) + A(t)u(t) &= \ell(t) & t \in J := (0, T), \\ u(0) &= u_0, \end{cases}$$

$$\underbrace{\iint_{J \times \Omega} \frac{du}{dt}(t, x) v(t, x) dt dx}_{(d_t u)(v) :=} + \underbrace{\int_J (A(t)(u(t)))(v(t)) dt}_{(\mathcal{A}u)(v) :=} = \underbrace{\iint_{J \times \Omega} \ell(t, x) v(t, x) dt dx}_{(\ell u)(v) :=}$$

- $X := L_2(J; V) \cap H^1(J; V')$, $Y := L_2(J; V)$, $B := d_t + \mathcal{A}: X \rightarrow Y'$.

- ▶ With $B_e := (B, \gamma_0)^\top$, solve $B_e u = (\ell, u_0)^\top$.

Theorem 9 (Zeidler ('90))

If $\mathcal{A}: Y \rightarrow Y'$ is monotone, hemicontinuous, coercive, and bounded, then $B_e: X \rightarrow Y' \times H$ is invertible.

Theorem 10

If $\mathcal{A}: Y \rightarrow Y'$ is Lipschitz continuous, and strongly monotone, then $B_e^{-1}: Y' \times H \rightarrow X$ is Lipschitz continuous.

Application: Quasi-linear parabolic PDE

- Generalize $\frac{du}{dt} - \operatorname{div}_x \nabla_x u$ to $\frac{du}{dt} - \operatorname{div}_x \mu(t, x, |\nabla_x u|^2) \nabla_x u$, i.e.,

$$(\mathcal{A}u)(v) := \iint_{J \times \Omega} \mu(t, x, |\nabla_x u|^2) \nabla_x u \cdot \nabla_x v \, dx \, dt$$

- [Zeidler ('90)]: If, for some $m_\mu, M_\mu > 0$,

$$m_\mu(r - s) \leq \mu(\cdot, \cdot, r^2)r - \mu(\cdot, \cdot, s^2)s \leq M_\mu(r - s) \quad (r \geq s \geq 0),$$

then $\mathcal{A}: Y \rightarrow Y'$ is **Lip. cont. & str. mon.**

Numerical approx.

- Minimal residual functional can have multiple local minima, so *not* suited.
- Inspired by Brézis & Ekeland ('76), Nayroles ('76), we consider

$$\begin{bmatrix} \mathcal{A} & \mathbf{d}_t \\ \mathbf{d}'_t & -(\mathcal{A} + \gamma'_T \gamma_T) \end{bmatrix} \begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} \ell \\ -(\ell + \gamma'_0 u_0) \end{bmatrix}. \quad (\text{BEN})$$

Map $Y \times X \rightarrow Y' \times X'$ is Lip., with Lip. inverse.

- $\mathbf{d}_t + \mathbf{d}'_t = \gamma'_T \gamma_T - \gamma'_0 \gamma_0 \rightsquigarrow \theta = u = B_e^{-1}(\ell, u_0)^\top$ (consistency).
- For $Y^\delta \subset Y$, $X^\delta \subset X$, (BEN^δ) has unique sol. $(\theta^\delta, u^\delta)$.[‡]
- If $X^\delta \subset Y^\delta$ and *if* $\gamma_\Delta > 0$, then

$$\|u - E_X^\delta u^\delta\|_X \lesssim \inf_{w \in X^\delta} \|u - E_X^\delta w\|_X \quad (\text{quasi-best approx.})$$

- Otherwise data-dependent optimality condition applies, which is enforced in double adaptivity loop.
- Nonlinear (BEN^δ) system solvable by Uzawa-Zarantonello iteration.

[‡]For $X^\delta \subset Y^\delta$, and $\mathcal{A}$ linear, with $\lambda^\delta := \theta^\delta - u^\delta$, $(\lambda^\delta, u^\delta)$ solves (MR^δ) !

Conclusion

- We've seen several space-time methods for parabolic problems that give quasi-optimal approximations (either unconditionally, or under uniform inf-sup condition, or by adaptive selection test space).
- Methods come with a posteriori error estimator, which induces an AFEM.
- Adaptivity improves rates for singular solutions. Optimal rates not (yet) demonstrated.
- Optimal preconditioners so far for tensor product trial spaces.
- Extension to parabolic problems with Lipschitz continuous and strongly monotone spatial PDOs.
- Solving problems in $d + 1$ dimensions requires more memory, but it can be advantageous for parallel computing.

Thank you for your attention / patience.

The End